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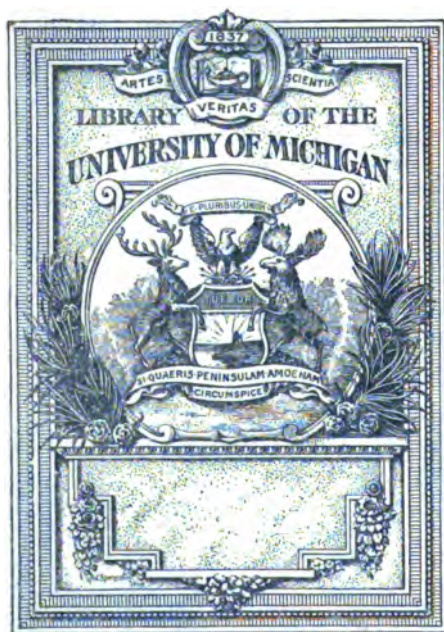
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**MATHEMATICAL  
CRYSTALLOGRAPHY**

**AND**

**THE THEORY OF GROUPS  
OF MOVEMENTS**

**BY**

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## PREFACE

THE object of this book is to collect for the use of

English readers those results of the mathematical theory of crystallography which are not proved in the modern textbooks on that subject in the English language. No claim to originality is made; the theorems proved are in most cases to be found in the works of Schoenflies, Liebisch, &c. From the scope of this treatise are excluded all practical applications of the results obtained (since these are explained at length in the existing English textbooks), and any attempt at a full description of the physical properties of crystals and of the phenomena connected with their growth. On the other hand, I have included a fairly full account of the geometrical theory of crystal-structure which the labours of Bravais, Jordan, Sohncke, Fedorow, Schoenflies, and Barlow have now completed. The last three authors have worked on somewhat different lines; I have in Part II of this book followed the methods and notation of Schoenflies very closely, since they lay the greatest stress on that isomorphism between point-group and space-group which is of such importance in the physical application of the purely geometrical theory. I cannot do better than refer the reader to Schoenflies' "Krystallsysteme und Krystallstruktur" where he will find at length what I have presented in a very

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MacLenn. M. P. 7-6-37

similar but shortened form. I have, however, adopted some features from the work of other authors; for instance, the method of chapter xx, §§ 15 and 16, is due to Barlow, the proof of chapter xvi, § 12, is an extension of one due to Jordan, &c. The figures illustrating the groups are on the lines suggested by Fedorow; they have, however, been drawn independently without reference to the diagrams in "Zeitschr. f. Kryst. u. Min." xxiv, which contain some errors. The figures in chapter xix, § 11, are due to Schoenflies, and figures 71, 72, 76, 90, 91, 181, 182, and 183 are also adapted from his "Krystall-systeme und Krystallstructur."

I have not attempted to give a complete list of references, but I have occasionally inserted them when lack of space prevented me from making more than a mere allusion to the contents of the paper quoted.

I take this opportunity of expressing my sincerest gratitude to Professor H. A. Miers, F.R.S., D.Sc., for his generous and sympathetic help. Without his assistance both in preparing the MS. and in correcting the proofs, the book could never have been written; and any merits it may have are due to his suggestions alone.

H. H.

*April* 16, 1903.

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# MATHEMATICAL CRYSTALLOGRAPHY

## PART I

### CHAPTER I

#### THE STEREOGRAPHIC PROJECTION.

§ 1. If we take any point  $O$ , and join it by straight lines to all the points of a figure  $U$ ; then the plane figure formed by the intersections of these straight lines with any plane  $p$  is called the projection of  $U$  on  $p$  with respect to  $O$ .

The projection of greatest interest in crystallography is the 'stereographic projection.' Before we consider this, however, we shall first prove some properties of 'inversion.'

§ 2. *Inversion.* If we take any point  $O$ , and if  $P$  be any point on a figure  $U$ , and we choose the point  $P'$  on the straight line  $OP$  such that  $OP \cdot OP' = k^2$ , where  $k$  is a constant; then the locus,  $U'$ , of the point  $P'$  is called the 'inverse' of  $U$  with respect to  $O$ .

The point  $P'$  is said to be the 'inverse' point of  $P$ .

If  $P, Q$  be two points very close together and  $P', Q'$  their inverse points; then  $PQ, P'Q'$  make equal angles with the line  $OPP'$ .

Because  $OP \cdot OP' = OQ \cdot OQ'$ ,  $\therefore P, P', Q, Q'$  are concyclic, and  $\therefore$  the angle  $OQP =$  the angle  $OP'Q'$ . But when  $Q$  is very

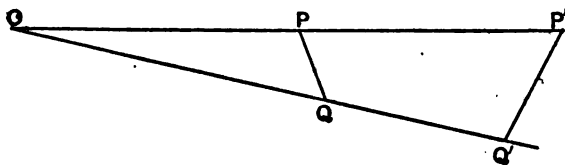


Fig. 1.

close to  $P$ , the angle  $OQP =$  the angle  $QPP'$ ;  $\therefore$  the angles  $QPP', Q'P'P$  are equal when  $Q$  is very close to  $P$ . If  $PQ$  is the tangent at  $P$  to a curve,  $P'Q'$  will be the tangent at  $P'$  to the inverse curve, hence we have

If  $P, P'$  are two corresponding points on inverse curves, the tangents at  $P$  and  $P'$  lie in the same plane and make equal angles with  $OPP'$ .

Any two curves cut at the same angle as the two inverse curves.

For let  $PA, PB$  be the tangents to two curves at their point of intersection  $P$ ; and let  $P'A, P'B$  be the tangents to the

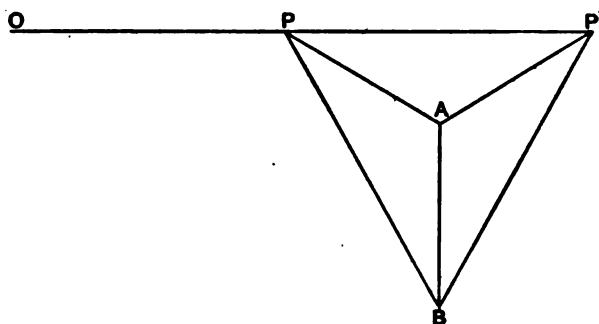


Fig. 2.

two inverse curves at the corresponding point  $P'$  (where the planes  $PAP', PBP'$  may or may not coincide).

Then  $\therefore$  the angle  $APP' =$  the angle  $AP'P$ ,  $\therefore AP = AP'$ ;

And  $\therefore$  the angle  $BPP' =$  the angle  $BP'P$ ,  $\therefore BP = BP'$ .

And  $AB$  is common to the triangles  $APB, AP'B$ ;  $\therefore$  the angle  $APB =$  the angle  $AP'B$ .

*The inverse of a sphere is a sphere or a plane.*

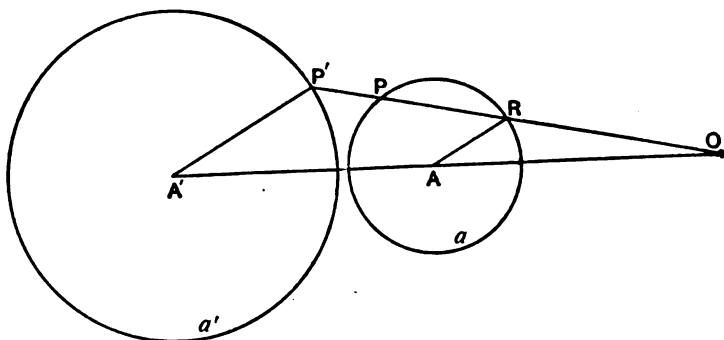


Fig. 3.

Let  $a$  be the sphere, and  $A$  its centre.

Let the length of the tangent from  $O$  to  $a$  be  $h$ . On  $OA$  take a point  $OA'$  such that  $OA' : OA :: h^2 : a^2$  (Fig. 3).

Let  $P$  and  $P'$  be inverse points,  $P$  being any point on  $a$ , and let  $OP$  meet  $a$  again in  $R$ .

Then since  $OP \cdot OP' = k^2$ ,  $OP \cdot OR = k^2$

$$\therefore OR : OP' :: k^2 : k^2 \\ :: OA : OA'.$$

Therefore the triangles  $OAR$ ,  $OA'P'$  are similar; and  $\therefore AR : A'P' :: OA : OA'$ .

But  $OA$ ,  $OA'$ ,  $AR$  are fixed in length, and  $\therefore$  so is  $A'P'$ ; hence, since  $A'$  is a fixed point,  $P'$  lies on a sphere whose centre is  $A'$ .

If  $O$  lies on  $a$  then  $A'$  is infinitely far from  $A$ , and the inverse of  $a$  is a sphere whose centre is infinitely distant from its surface; i.e. the inverse of  $a$  is a plane.

Conversely, if  $a'$ ,  $a$  are any two spheres, a point can be found with respect to which  $a'$ ,  $a$  are inverse to one another.

For (Fig. 3) let  $A'$ ,  $A$  be the centres of  $a'$ ,  $a$ , and let  $A'P'$ ,  $AR$  be any two parallel radii, and let  $P'R$  meet  $A'A$  in  $O$ , and  $a$  again in  $P$ .

Then  $OA : OA' :: AR : A'P'$ , and  $\therefore O$  is a fixed point. Also since  $OR \cdot OP$  is constant, and the ratio  $OR : OP' = OA : OA'$  which is constant;  $\therefore OP \cdot OP'$  is constant.

Again, if  $p$  is a plane, and  $a$  a sphere;  $a$  and  $p$  are inverse with respect to either point in which a perpendicular to  $p$  through the centre of  $a$  meets  $a$ .

For (Fig. 4), let  $A$  be the centre of  $a$ , and draw  $OAC'C$  perpendicular to  $p$ , meeting  $a$  at  $C'$  and  $O$ , and  $p$  at  $C$ . Take any point  $P$  on  $p$ , and join  $OP'P$  cutting  $a$  at  $P'$  and  $O$ .

Then  $\therefore$  the angles  $C'CP$ ,  $C'PP$  are right angles; so that  $P$ ,  $C$ ,  $C'$ ,  $P'$  are concyclic;  $\therefore OP' \cdot OP = OC' \cdot OC$  which is constant.

Similarly we may prove that  $a$  and  $p$  are inverse with respect to  $C'$ .

Since the inverse of the intersection of two surfaces is evidently the intersection of the two inverse surfaces, and since the intersection of two spheres is a circle, we see at once that

*The inverse of a circle is a circle.*

§ 3. *The Stereographic projection.* In this method of projection figures lying on the surface of a sphere are projected from a point  $O$  also lying on the surface of the sphere, on to any plane  $p$  parallel to the tangent plane at  $O$  (Fig. 4).

The projected figure is the inverse of the original with respect to  $O$  (§ 2); hence a circle projects into a circle (or



$e, f$ , then any two planes through  $E$  and  $F$  cut off equal arcs from  $e$  and  $f$  (Fig. 6).

For let the planes cut  $e$  in  $U, U'$  and  $f$  in  $V, V'$  respectively; and let the great circle through  $E$  and  $F$  cut  $e$  and  $f$  in  $X$  and  $Y$ . Then the arc  $EU$

$$= \frac{\pi}{2} = \text{the arc } FV;$$

$\therefore$  the straight line  $EU$  = the straight line  $FV$ .

Now in the plane triangles  $EUF, FVE$ ,  $EU = FV$ ,  $EF$  is common, and the angle  $EUF =$  the angle  $FVE$  (for  $E, V, U, F$  lie on a small circle);  $\therefore$  the angle  $UEF =$  the angle  $VFE$ \* and  $FU = EV$ , and  $\therefore$  the arc  $FU =$  the arc  $EV$ . Again, since in the spherical

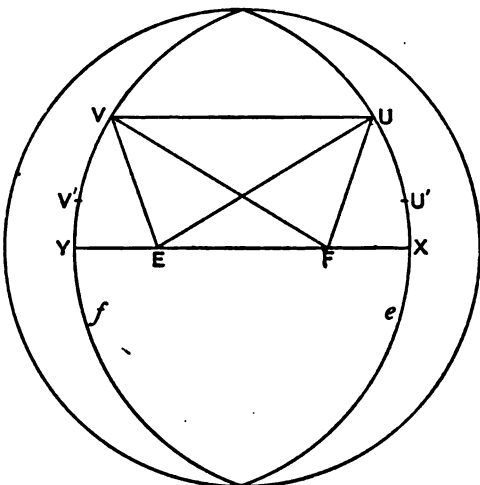


Fig. 6.

triangles  $EUF, FVE$ ,  $EU = FV$ ,  $FU = EV$ , and  $EF$  is common; therefore the (spherical) angle  $FEU =$  the angle  $EFV$ .

Moreover, since in the spherical triangles  $UEX, VFY$ ,  $EU = \frac{\pi}{2} = FV$ ,  $EX = \frac{\pi}{2} = FY$ , and the angle  $UEX =$  the angle  $VFY$ ;  $\therefore$  the arcs  $UX, VY$  are equal.

Similarly the arcs  $U'X, V'Y$  are equal.

Hence the arcs  $UU', VV'$  are equal.

Now the projection of any diameter of  $s'$  on the plane  $p$  with respect to the point  $O$  is a diameter of the circle  $s$ . But any great circle meets  $s'$  at the ends of a diameter; therefore the projection of any great circle meets  $s$  at the ends of a diameter.

Consider any two points  $P, Q$  on a diameter of  $s$ . Let  $D, S$  be the ends of the diameter perpendicular to  $PQ$ .

\* Since the plane  $EFUV$  cuts each of the circles  $e$  and  $f$  in two points, the pair of points  $U$  and  $V$  may be taken in four different ways. We have taken them in the figure so that  $EV$  and  $FU$  are not parallel, and therefore the angles  $UFE, VEF$  are not supplementary. Then it follows from the reasoning in the text that they are equal; and hence that the angles  $UEF, VFE$  are equal.

Join  $DP$ ,  $DQ$  and produce them to meet  $s$  again in  $P_1$ ,  $Q_1$  (Fig. 7).

Let  $P'$ ,  $Q'$ ,  $D'$ ,  $S'$ ,  $P_1'$ ,  $Q_1'$  be the points of which  $P$ ,  $Q$ ,  $D$ ,  $S$ ,  $P_1$ ,  $Q_1$  are the projections. Then

$D'$  is the pole of the great circle through  $P'$  and  $Q'$ , and  $O$  is the pole of the great circle  $D'P_1'S'$  (i.e. the circle  $s'$ ).

Hence, by the above, the planes  $D'P'O$ ,  $D'Q'O$  (which are the same as  $DPO$ ,  $DQO$ ) cut off equal arcs  $P'Q'$ ,  $P_1'Q_1'$ ; i.e.  $PQ$ ,  $P_1Q_1$  represent equal arcs.

Again, evidently  $P_1Q_1$  sub-

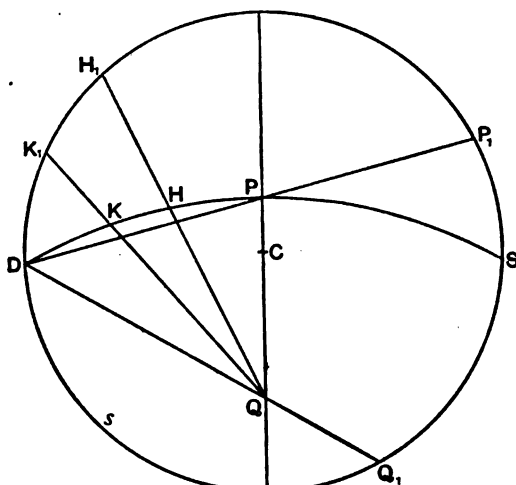


Fig. 7.

tends the same angle at  $C$  as  $P_1'Q_1'$  does at the centre of the sphere.

Hence  $PQ$  is the projection of an arc subtending an angle  $P_1CQ_1$  at the centre of the sphere.

In particular, if  $P_1CQ_1$  is a right angle,  $Q$  represents the pole of the circle whose projection is the arc  $DPS$ .

Again, draw any two straight lines through  $Q$  meeting  $DPS$ ,  $DP_1S$  at  $K$ ,  $H$  and  $K_1$ ,  $H_1$  respectively.

If  $D'$ ,  $P'$ ,  $S'$ ,  $Q'$ ,  $K'$ ,  $H'$ ,  $K_1'$ ,  $H_1'$  are the points of which  $D$ ,  $P$ ,  $S$ ,  $Q$ ,  $K$ ,  $H$ ,  $K_1$ ,  $H_1$  are the projections; and if  $P_1CQ_1$  is a right angle, then  $Q'$  is the pole of  $D'P'S'$ . Also  $O$  is the pole of  $D'P_1'S'$ ; and hence the planes  $Q'OK'$ ,  $Q'OH'$  (which are the same as  $QOK$ ,  $QOH$ ) cut off equal arcs  $H'K'$ ,  $H_1'K_1'$ ; i.e.  $HK$  and  $H_1K_1$  represent equal arcs. As before, the angle  $K_1CH_1$  = the angle subtended by  $H_1'K_1'$  at the centre of the sphere; and therefore  $HK$  is the projection of an arc subtending an angle  $H_1CK_1$  at the centre of the sphere.

§ 5. Hence we have now shown how to find the projection of that arc of any great circle which subtends any given angle at the centre of the sphere.

Suppose for instance we are given the circle  $DPS$  of the projection, and a point  $H$  on it (Fig. 7), and we wish to find

a point  $K$  on the circle such that the arc  $HK$  is the projection of an arc subtending an angle  $a$  at the centre of the sphere; then we proceed as follows:—

Bisect  $DS$  at  $P$ , produce  $DP$  to meet  $s$  again in  $P_1$ . Take the point  $Q_1$  on  $s$ , such that  $P_1CQ_1$  is a right angle; join  $DQ_1$  cutting  $PC$  at  $Q$ .

Then  $Q$  represents the pole of the circle whose projection is  $DPS$ .

Produce  $QH$  to meet  $s$  at  $H_1$ ; take  $K_1$  on  $s$  such that the angle  $H_1CK_1 = a$ . Join  $QK_1$  cutting  $DPS$  at  $K$ . Then  $K$  is the point required.

## CHAPTER II

PROPERTIES COMMON TO SYMMETRICAL AND  
ASYMMETRICAL CRYSTALS.

§ 1. *A portion of solid matter is said to be homogeneous if its physical and chemical properties are the same about every point of its substance\*.*

Solid homogeneous matter is either amorphous or crystalline.

*Solid homogeneous matter is amorphous if its physical properties are the same in every direction; if this is not so the matter is said to be crystalline.*

Thus if rods of given length and cross section are cut in various directions from the matter; and if *all* the rods conduct heat and electricity with equal ease, offer an equal resistance to compression, &c., the matter is amorphous; if otherwise it is crystalline: but in either case rods cut in the same direction from different parts of the matter have identical properties, because the matter is homogeneous.

When solid crystalline material is formed in an unconfined space, either from the molten substance, from a vapour, or from a solution, it is found that the material is bounded by plane surfaces forming a polyhedron. This polyhedron is called a *crystal* of the substance; the planes bounding it are called the *faces* of the crystal; and the intersection of any two such planes is called an *edge* of the crystal.

Though the appearance of the crystal may vary very much with the conditions of growth, it is found that the angles between its faces are constant for a given substance, and are characteristic of that substance.

If from a fixed point we draw perpendiculars to all the faces of any crystal of a given substance, we obtain a pencil of lines which is characteristic of that substance.

§ 2. Let planes through a point *O* within the crystal parallel

\* With the exception of points whose distance from the boundary of the substance is comparable with the size of the molecules of which the matter is formed.

to any three faces of the crystal meet in the straight lines  $OA, OB, OC$ ; and let any other face (called the *parametral* face) meet these lines in  $A, B, C$ . Suppose that  $OA:OB:OC = a:b:c$ .

Let any other face of the crystal cut  $OA, OB, OC$  in  $A', B', C'$ ; and let  $OA':OB':OC' = \frac{a}{h}:\frac{b}{k}:\frac{c}{l}$ .

Then  $h, k, l$  are said to be the *indices* of the face  $A'B'C'$ ; the face is represented by the symbol  $(hkl)$  and is known as 'the face  $(hkl)$ .'

This is Miller's notation, and is now universally adopted. We take  $h$  positive if  $A'$  is on the same side of  $O$  as  $A$ , otherwise negative; and similarly for  $k$  and  $l$ .

If  $h = 0$  the face  $A'B'C'$  is parallel to  $OA$ , and similarly for  $k$  and  $l$ .

At present we shall only concern ourselves with the *ratios*  $h:k:l$  so long as  $h, k$ , and  $l$  do not alter their signs; thus  $(hkl)$ ,  $(xh\ xk\ xl)$  represent the same face if  $x$  is positive. A distinction is made, however, between two parallel faces on opposite sides of  $O$ ; thus  $(hkl)$ ,  $(-h\ -k\ -l)$  represent different (but parallel) faces.

It is usual to take for  $h, k, l$  quantities as simple as possible; thus we talk of 'the face  $(212)$ ,' not of 'the face  $(636)$ ,' or the 'face  $(\sqrt{3}\ \frac{\sqrt{3}}{2}\ \sqrt{3})$ .' We shall see later that it is always possible to express  $h, k, l$  as small integers.

In the symbols which represent faces it is customary to print minus signs *above* instead of *before* the letter or number; thus for  $(-h\ -k\ -l)$  we write  $(\bar{h}\ \bar{k}\ \bar{l})$ .

A face parallel to  $OBC$  is represented by  $(100)$  or  $(\bar{1}00)$ , so for faces parallel to  $OCA$  and  $OAB$ .

Again, we are only concerned with the *ratios*  $a:b:c$ . It is usual to take  $b = 1$ . These ratios are called the *axial ratios*. The lines  $OA, OB, OC$  are called the *crystallographic axes*; they are parallel to 3 crystal edges.

§ 3. Given the indices of a face referred to one set of crystallographic axes through  $O$ , to find the indices referred to any other set of crystallographic axes through  $O$ .

Let  $(x' y' z')$  be the Cartesian coordinates of any point referred to the first set of crystallographic axes, and  $(X Y Z)$  the Cartesian coordinates of the same point referred to the new crystallographic axes.

Let  $x' = ax, y' = by, z' = cz$ .

Now  $X, Y, Z$  are connected with  $x', y', z'$  and therefore with  $x, y, z$  by linear relations\*;

Let these relations be

$$\left. \begin{aligned} x &= l_1 X + m_1 Y + n_1 Z \\ y &= l_2 X + m_2 Y + n_2 Z \\ z &= l_3 X + m_3 Y + n_3 Z \end{aligned} \right\}.$$

Let

$$\left. \begin{aligned} d_1 x + e_1 y + f_1 z &= 0 \\ d_2 x + e_2 y + f_2 z &= 0 \\ d_3 x + e_3 y + f_3 z &= 0 \end{aligned} \right\}$$

be the equations referred to the old axes of the three planes whose intersections form the new axes; and let  $D_1, E_1, \dots$  be the co-factors of  $d_1, e_1, \dots$  in the determinant

$$\begin{vmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{vmatrix}.$$

Then the intersection of

$$\left. \begin{aligned} d_1 x + e_1 y + f_1 z &= 0 \\ d_2 x + e_2 y + f_2 z &= 0 \end{aligned} \right\}; \text{ i. e. } \frac{x}{D_1} = \frac{y}{E_1} = \frac{z}{F_1},$$

$$\text{or } \frac{l_1 X + m_1 Y + n_1 Z}{D_1} = \frac{l_2 X + m_2 Y + n_2 Z}{E_1} = \frac{l_3 X + m_3 Y + n_3 Z}{F_1}$$

must be identical with  $Y = Z = 0$ .

$$\text{Therefore } \frac{l_1}{D_1} = \frac{l_2}{E_1} = \frac{l_3}{F_1} = \frac{1}{P} \text{ (say); and similarly}$$

$$\frac{m_1}{D_2} = \frac{m_2}{E_2} = \frac{m_3}{F_2} = \frac{1}{Q} \text{ (say); and } \frac{n_1}{D_3} = \frac{n_2}{E_3} = \frac{n_3}{F_3} = \frac{1}{R} \text{ (say).}$$

Let  $(def)$  be the symbol of any face referred to the old axes; the equation of a parallel plane through  $O$  is

$$\frac{x'}{\frac{a}{d}} + \frac{y'}{\frac{b}{e}} + \frac{z'}{\frac{c}{f}} = 0,$$

or

$$dx + ey + fz = 0.$$

Its equation referred to the new axes is therefore

$$d(l_1 X + m_1 Y + n_1 Z) + e(l_2 X + m_2 Y + n_2 Z) + f(l_3 X + m_3 Y + n_3 Z) = 0,$$

$$\text{or } X(dl_1 + el_2 + fl_3) + Y(dm_1 + em_2 + fm_3) + Z(dn_1 + en_2 + fn_3) = 0,$$

$$\text{or } \frac{X}{P}(dD_1 + eE_1 + fF_1) + \frac{Y}{Q}(dD_2 + eE_2 + fF_2) + \frac{Z}{R}(dD_3 + eE_3 + fF_3) = 0.$$

\* The nature of these relations is discussed in chapter vii.

Hence if we take  $P:Q:R$  as our new axial ratios we must take

$$(dD_1 + eE_1 + fF_1, dD_2 + eE_2 + fF_2, dD_3 + eE_3 + fF_3)$$

as the new symbol of the face ( $d\ e\ f$ ).

To obtain  $P:Q:R$  we proceed as follows:—

Let  $\alpha, \beta, \gamma$  be the angles the old axes make with each other. The square of the distance of any point on the line  $Y = Z = 0$  from  $O$ , is  $X^2$ , and is also

$$\begin{aligned} & x^2 + y^2 + z^2 + 2yz' \cos \alpha + 2z'x' \cos \beta + 2x'y' \cos \gamma \\ &= X^2 (a^2 l_1^2 + b^2 l_2^2 + c^2 l_3^2 + 2bcl_2 l_3 \cos \alpha + 2cal_3 l_1 \cos \beta \\ & \quad + 2abl_1 l_2 \cos \gamma) \\ &= \frac{X^2}{P^2} (a^2 D_1^2 + b^2 E_1^2 + c^2 F_1^2 + 2bcE_1 F_1 \cos \alpha + 2caF_1 D_1 \cos \beta \\ & \quad + 2abD_1 E_1 \cos \gamma) \end{aligned}$$

$$\therefore P^2 = (a^2 D_1^2 + b^2 E_1^2 + c^2 F_1^2 + 2bcE_1 F_1 \cos \alpha + 2caF_1 D_1 \cos \beta + 2abD_1 E_1 \cos \gamma);$$

and similar expressions hold for  $Q^2$  and  $R^2$ .

§ 4. It is an observed fact that, for *any* face ( $d\ e\ f$ ) of a crystal, the ratios  $d:e:f$  are *rational*; and that in general  $d, e, f$  are in the ratios of *small* integers, e. g. the faces liable to occur are  $(1\ 1\ 1)$ ,  $(1\ \bar{2}\ 2)$ ,  $(3\ 2\ 1)$ ,  $(4\ 2\ \bar{1})$  and so on.

This 'law' is called the *law of rational indices*; and it is on this law combined with the constancy of angles in different crystals of a given substance, that the whole of the argument of the first part of this book is based.

If the law holds for one set of crystallographic axes it holds for any other.

For the ratios  $d:e:f$ ;  $d_1:e_1:f_1$ ;  $d_2:e_2:f_2$ ;  $d_3:e_3:f_3$  of the previous section are all rational; hence, since we are only concerned with the *ratios* of  $d, e, f$ , &c. (cf. p. 9), we may take  $d, e, f, d_1, e_1, f_1, d_2, e_2, f_2, d_3, e_3, f_3$  themselves rational. In this case  $D_1, D_2, D_3, E_1, E_2, E_3, F_1, F_2, F_3$  are also rational, and therefore  $dD_1 + eE_1 + fF_1, dD_2 + eE_2 + fF_2, dD_3 + eE_3 + fF_3$  are also rational.

We may sum up the above statements thus:—

If ( $h\ k\ l$ ) be any face of a crystal, referred to lines through a point parallel to any three edges as crystallographic axes; then  $h:k:l = m:n:p$ , where  $m, n, p$  are integers and in general small integers.

All that is meant is that no measurements which we can make can prove that the indices of any given crystal face are not in the ratios of small integers. The most convenient

and accurate means we possess of finding the indices of the faces of a crystal is that of measuring the angles between the various faces, and all such measurements are liable to a certain range of error. Hence to say merely that 'we cannot prove that the indices of any given face are not in the ratios of integers' would be only equivalent to saying that, 'if we are given any incommensurable quantity  $r$ , we can always find a commensurable quantity  $s$  such that  $r-s$  is as small as we please'; but to say that 'we cannot prove that the indices of any given face are not in the ratios of *small* indices' is to state an experimental law of the highest importance.

We pointed out before that we are only concerned with the *ratios* of the indices of any face; we now see that it is always possible to take the indices as integers. We shall in future take as the indices of any face the *smallest integers possible*, and talk for instance of 'the face  $(1\ 2\ 3)$ ,' not of 'the face  $(2\ 4\ 6)$ ,' or 'the face  $(\frac{1}{2}\ \frac{2}{3}\ 1)$ .'

§ 5. If (as in § 3) the Cartesian coordinates of any point referred to the crystallographic axes are  $(x' y' z')$ , and if  $x' = ax$ ,  $y' = by$ ,  $z' = cz$ ; then the equations of the planes through the origin  $O$  parallel to the faces  $(h_1 k_1 l_1)$  and  $(h_2 k_2 l_2)$  are

$$h_1 x + k_1 y + l_1 z = 0$$

$$h_2 x + k_2 y + l_2 z = 0.$$

Therefore the equation of the line through  $O$  parallel to the edge in which the two faces meet is

$$\frac{x}{k_1 l_2 - k_2 l_1} = \frac{y}{l_1 h_2 - l_2 h_1} = \frac{z}{h_1 k_2 - h_2 k_1}.$$

The quantities  $k_1 l_2 - k_2 l_1$ ,  $l_1 h_2 - l_2 h_1$ ,  $h_1 k_2 - h_2 k_1$  are integral since  $h_1, k_1, l_1, h_2, k_2, l_2$  are all integral; the smallest integers proportional to them are called the *indices of the edge*\* in which the two faces meet.

If  $H, K, L$  are the indices of any edge, it is represented by the symbol  $[H\ K\ L]$  and is known as 'the edge  $[H\ K\ L]$ .'

The direction-ratios of the edge  $[1\ 1\ 1]$  are evidently proportional to  $a, b, c$ .

If  $[H_1\ K_1\ L_1]$ ,  $[H_2\ K_2\ L_2]$  are any two edges, the equation of the plane through  $O$  parallel to both edges is

$$(K_1 L_2 - K_2 L_1)x + (L_1 H_2 - L_2 H_1)y + (H_1 K_2 - H_2 K_1)z = 0.$$

This is parallel to the face whose indices are  $K_1 L_2 - K_2 L_1$ ,

\* The reader must carefully distinguish between 'indices of a face' and 'indices of an edge.'

$L_1H_3 - L_2H_1, H_1K_3 - H_2K_1$  \*: that is, to a face whose indices are integral, since  $H_1, K_1, L_1, H_2, K_2, L_2$  are all integral.

We have then,

*The intersection of any two possible crystal faces is a possible crystal edge, and a plane parallel to any two possible crystal edges is a possible crystal face,*

meaning by a 'possible' crystal face or edge one whose indices are integral.

§ 6. Crystal faces which are all parallel to one edge are said to form a *zone*. A line through  $O$  parallel to the edge is called the *zonal axis*; the normals through any point to the faces of a zone evidently lie in a plane perpendicular to the zonal axis.

The equations of planes through  $O$  parallel to the faces

$(h_1k_1l_1), (h_2k_2l_2), (h_3k_3l_3), (h_4k_4l_4) \dots$  are

$$h_1x + k_1y + l_1z = 0, h_2x + k_2y + l_2z = 0,$$

$$h_3x + k_3y + l_3z = 0, h_4x + k_4y + l_4z = 0, \&c.;$$

and hence the condition that these faces should form a zone is that all the determinants represented by

$$\begin{vmatrix} h_1 & h_2 & h_3 & h_4 & \cdot & \cdot \\ k_1 & k_2 & k_3 & k_4 & \cdot & \cdot \\ l_1 & l_2 & l_3 & l_4 & \cdot & \cdot \end{vmatrix}$$

should be equal to zero.

Taking the three sets of indices  $h_1k_1l_1, h_2k_2l_2, h_nk_nl_n$ , we have

$$\begin{vmatrix} h_1 & h_2 & h_n \\ k_1 & k_2 & k_n \\ l_1 & l_2 & l_n \end{vmatrix} = 0$$

and  $\therefore$

$$\begin{vmatrix} h_1 & h_2 & h_n - \lambda h_1 - \mu h_2 \\ k_1 & k_2 & k_n - \lambda k_1 - \mu k_2 \\ l_1 & l_2 & l_n - \lambda l_1 - \mu l_2 \end{vmatrix} = 0.$$

Not all the quantities  $k_1l_2 - k_2l_1, l_1h_2 - l_2h_1, h_1k_2 - h_2k_1$ , can = 0, for that would involve

$$\frac{h_1}{h_2} = \frac{k_1}{k_2} = \frac{l_1}{l_2};$$

suppose the first of them  $\neq 0$ , and choose  $\lambda, \mu$  to satisfy the equations  $k_n - \lambda k_1 - \mu k_2 = l_n - \lambda l_1 - \mu l_2 = 0$ .

Then the determinant last written is equal to

$$\begin{vmatrix} k_1 & k_2 \\ l_1 & l_2 \end{vmatrix} \times (h_n - \lambda h_1 - \mu h_2), \text{ and } \therefore h_n - \lambda h_1 - \mu h_2 = 0;$$

for  $\begin{vmatrix} k_1 & k_2 \\ l_1 & l_2 \end{vmatrix} \neq 0$ .

\* Or smaller integers proportional to these quantities.

Hence the indices of any face of the zone to which the two faces  $(h_1 k_1 l_1)$   $(h_2 k_2 l_2)$  belong are of the form

$$(\lambda h_1 + \mu h_2, \lambda k_1 + \mu k_2, \lambda l_1 + \mu l_2).$$

The converse is also true since

$$\begin{vmatrix} h_1 & h_2 & \lambda h_1 + \mu h_2 \\ k_1 & k_2 & \lambda k_1 + \mu k_2 \\ l_1 & l_2 & \lambda l_1 + \mu l_2 \end{vmatrix} \equiv 0.$$

§ 7. To find the anharmonic ratio of four faces of a zone.

Let  $n_1, n_2, n_3, n_4$  be the four faces; and let  $[n_i n_j]$  denote the angle between  $n_i$  and  $n_j$  (where  $i$  and  $j$  are any two of the integers 1, 2, 3, 4)\*. Then

$$\sin [n_1 n_3] \cdot \sin [n_2 n_4] - \sin [n_1 n_4] \cdot \sin [n_2 n_3]$$

is called the anharmonic or cross ratio of the four faces.

Let perpendiculars  $VN_1, VN_2, VN_3, VN_4$  be drawn from any point  $V$  to the faces; then evidently the anharmonic ratio of the faces =  $\sin N_3 VN_1 \cdot \sin N_4 VN_2 - \sin N_4 VN_1 \cdot \sin N_3 VN_2$ , the anharmonic ratio of the pencil  $V (N_1 N_2 N_3 N_4)$ †. We denote this pencil by  $[P]$ ; its anharmonic ratio is evidently independent of the position of  $N$ .

Draw four planes through  $V$  perpendicular to the four lines of the pencil  $[P]$ ; these are parallel to the crystal faces and meet in a line (the line through  $V$  parallel to the zonal axis).

These four planes are cut by the plane of the pencil  $[P]$  in a pencil having the same angles, and therefore the same anharmonic ratio as the pencil  $[P]$ . Now it is a well-known theorem that the pencil in which four planes through a line are cut by *any* plane has a constant anharmonic ratio; hence the anharmonic ratio of the pencil  $[P]$  is equal to the anharmonic ratio of the pencil in which four planes through  $V$  parallel to the crystal faces are cut by any plane  $\Pi$ .

As before, let  $OA, OB, OC$  be the crystallographic axes, and  $a:b:c$  the axial ratios. We noticed above that the anharmonic ratio of the pencil  $[P]$  is independent of the position of the vertex  $V$ , we may therefore take  $V$  on the axis  $OB$ , and take the plane  $OBC$  as the plane  $\Pi$ .

Let  $h_1 k_1 l_1, h_2 k_2 l_2, h_3 k_3 l_3, h_4 k_4 l_4$  be the indices of the four faces; their intercepts on the axes are proportional to

$$\frac{a}{h_1} \frac{b}{k_1} \frac{c}{l_1}, \text{ \&c.}$$

\* This angle is the external angle between the two faces, so that  $[n_i n_j] = 0$  if  $n_i$  and  $n_j$  coincide.

† This is the definition of anharmonic ratio adopted by crystallographers; it is not quite the same as that commonly used by mathematicians.

Let  $OC$  cut the four planes through  $V$  parallel to the crystal faces in  $X_1, X_2, X_3, X_4$ . Then the pencil  $[P]$  has the same anharmonic ratio as the pencil  $V(X_1 X_2 X_3 X_4)$ ; i. e. as the range  $(X_1 X_2 X_3 X_4)$ . Therefore the anharmonic ratio of the pencil  $[P]$

$$= \frac{(OX_3 - OX_1)(OX_4 - OX_2)}{(OX_4 - OX_1)(OX_3 - OX_2)}.$$

Now

$$\frac{OX_1}{OV} = \frac{\frac{c}{l_1}}{\frac{b}{k_1}} = \frac{ck_1}{bl_1}, \text{ \&c.};$$

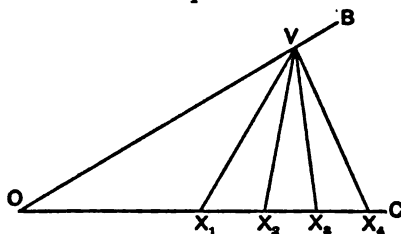


Fig. 8.

$$\text{hence } OX_1 = OV \cdot \frac{ck_1}{bl_1}; \quad OX_2 = OV \cdot \frac{ck_2}{bl_2};$$

$$OX_3 = OV \cdot \frac{ck_3}{bl_3}; \quad OX_4 = OV \cdot \frac{ck_4}{bl_4}.$$

Therefore the anharmonic ratio of the four faces = the anharmonic ratio of the pencil  $[P]$ ,

$$\begin{aligned} &= \frac{\left(\frac{ck_3}{bl_3} - \frac{ck_1}{bl_1}\right) \left(\frac{ck_4}{bl_4} - \frac{ck_2}{bl_2}\right)}{\left(\frac{ck_4}{bl_4} - \frac{ck_1}{bl_1}\right) \left(\frac{ck_3}{bl_3} - \frac{ck_2}{bl_2}\right)} = \frac{(k_1 l_3 - k_3 l_1)(k_2 l_4 - k_4 l_2)}{(k_1 l_4 - k_4 l_1)(k_2 l_3 - k_3 l_2)} \\ &= (\text{similarly}) \frac{(l_1 h_3 - l_3 h_1)(l_2 h_4 - l_4 h_2)}{(l_1 h_4 - l_4 h_1)(l_2 h_3 - l_3 h_2)} \\ &= \frac{(h_1 k_3 - h_3 k_1)(h_2 k_4 - h_4 k_2)}{(h_1 k_4 - h_4 k_1)(h_2 k_3 - h_3 k_2)} * . \end{aligned}$$

Since  $h_1, k_1, l_1$ , &c., are all integral, therefore

*the anharmonic ratio of four faces of a zone is rational †.*

\* For other proofs see Maskelyne's "Morphology of Crystals," pp. 70 and 71; Lewis's "Crystallography," p. 81 (G. Césaro, Rivista di Min. e Crist. Ital., 1889); Liebisch's "Grundriss der physikalischen Krystallographie," pp. 81 and 82, &c. The above proof was given by Hilary Bauerman ("Systematic Mineralogy," 1881, p. 35), and independently by the author ("Min. Mag." XIII, 1900, p. 69).

† This theorem is due to C. Fr. Gauss, 1831.

If the indices of the edge in which  $n_i$  and  $n_j$  meet are  $H_{ij}$ ,  $K_{ij}$ ,  $L_{ij}$  we see that

$$\frac{H_{13} \cdot H_{24}}{H_{14} \cdot H_{23}} = \frac{K_{13} \cdot K_{24}}{K_{14} \cdot K_{23}} = \frac{L_{13} \cdot L_{24}}{L_{14} \cdot L_{23}}.$$

§ 8. The easiest and most accurate way of determining the geometrical properties of a given crystal, its axial ratios, and the indices of all its faces is by measuring the angles between its faces. To do this we use an instrument called a *goniometer* (Fig. 9). It usually consists of a fixed collimator  $C$  with an illuminated slit  $S$ ; a fixed telescope  $T$ , the centre

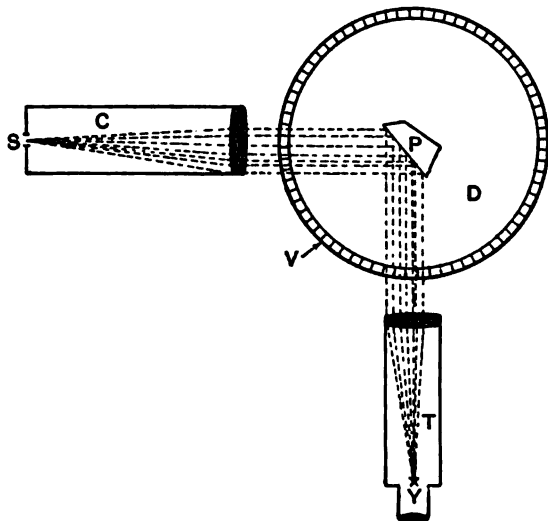


Fig. 9.

of whose field of view is marked with cross wires  $Y$ ; and a divided circle  $D$  attached to a crystal-holder which can be rotated about an axis  $a$  perpendicular to the plane of the divided circle, and passing through the centre of this circle. The axes of the collimator and telescope intersect at a point  $P$  on the axis  $a$ , and are perpendicular to this axis. The crystal is fixed to the holder and is adjusted so that  $P$  is its middle point, and so that two of its faces (and therefore the edge in which they meet) are parallel to the axis  $a$ . On rotating the crystal about this axis the image of the slit  $S$  will be seen in the telescope by reflexion at each of these faces in turn. The circle and crystal are firmly clamped together, and turned till the image of the slit is seen in the centre of the field of

view of the telescope by reflexion at one of the faces. A reading of the circle is then taken by means of a fixed pointer (or vernier)  $V$ . The crystal and circle are then turned till the image of the slit is seen by reflexion at the other face, and another reading is taken. The difference between the two readings evidently gives the external angle between the two faces.

The angles between all the faces of one zone can be measured without readjusting the crystal.

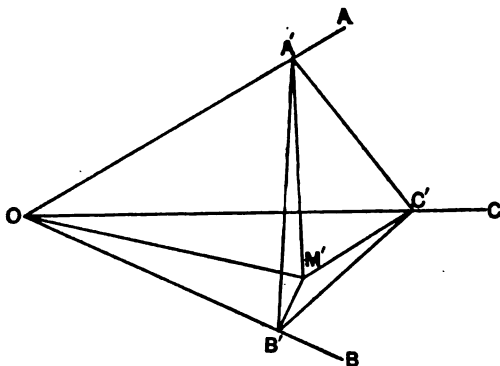


Fig. 10.

§ 9. We are at liberty to choose lines  $OA, OB, OC$  parallel to any three edges of a crystal (no two of which are parallel) as crystallographic axes, and the indices of any face ( $m$ ) not parallel to  $OA, OB$  or  $OC$  as  $(111)$ . The indices of faces parallel to  $OBC, OCA, OAB$  are then  $(100), (010), (001)$  respectively and  $m$  is the 'parametral face.'

We now proceed to show how the axial ratios, and the indices of the faces may be calculated from observations (made with the goniometer) of the angles between various faces.

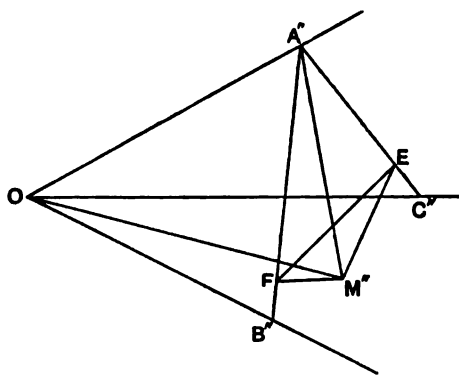


Fig. 11.

Let any plane parallel to  $m$ , cut  $OA, OB, OC$  and the normal to  $m$  through  $O$ , in  $A', B', C', M'$  (Fig. 10), then  $a:b:c$  (the axial ratios)

$$= OA' : OB' : OC' = \frac{OA'}{OM'} : \frac{OB'}{OM'} : \frac{OC'}{OM'}$$

$= \sec M'OA' : \sec M'OB' : \sec M'OC'$ , for the angles  $OM'A', OM'B', OM'C'$  are right angles.

Again, let any plane perpendicular to  $OA$  meet  $OA$ ,  $OB$ ,  $OC$  and the line through  $O$  normal to  $m$  in  $A''$ ,  $B''$ ,  $C''$ , and  $M''$  (Fig. 11). Draw  $M''E$ ,  $M''F$  perpendicular to  $A''C''$  and  $A''B''$  respectively.

Then because  $OA''$  is perpendicular to the plane  $A''EM''F$ , therefore  $OA''$  is perpendicular to  $M''E$ ,  $M''F$ .

Hence  $M''E$  is perpendicular to  $A''C''$  and  $OA''$ , and is therefore perpendicular to the plane  $OA''C''$ ; similarly  $M''F$  is perpendicular to the plane  $OA''B''$ .

Let the normal to  $m$  make angles  $\lambda$ ,  $\mu$ ,  $\nu$  with the normals to  $OBC$ ,  $OCA$ ,  $OAB$ ; these angles are measured with the goniometer.

Then  $OM''E = \pi - \mu$ ,  $OM''F = \pi - \nu$ .

Again, let the normals to  $OBC$ ,  $OCA$ ,  $OAB$  make angles  $A$ ,  $B$ ,  $C$ \* with each other; then evidently

$$\pi - FA''E = FM''E = A.$$

Since  $A''$ ,  $F$ ,  $M''$ ,  $E$  lie on a circle of which  $A''M''$  is a diameter, therefore  $\frac{A''M''}{OM''} = \frac{FE}{\sin A} \cdot \frac{1}{OM''}$

$$= \sqrt{\frac{M''E^2 + M''F^2 - 2M''E \cdot M''F \cdot \cos A}{\sin^2 A \cdot OM''^2}}.$$

Hence, since  $OA''M''$ ,  $OFM''$ ,  $OFM''$  are right angles,

$$\sin A''OM'' = \sqrt{\frac{\cos^2 \mu + \cos^2 \nu - 2 \cos \mu \cdot \cos \nu \cdot \cos A}{\sin^2 A}},$$

and consequently  $\sec A''OM'' = \sec A'OM'$  (Fig. 10)

$$= \frac{\sin A}{\sqrt{\sin^2 A - \cos^2 \mu - \cos^2 \nu + 2 \cos \mu \cdot \cos \nu \cdot \cos A}}.$$

Therefore  $a : b : c$

$$\begin{aligned} &= \frac{\sin A}{\sqrt{\sin^2 A - \cos^2 \mu - \cos^2 \nu + 2 \cos \mu \cdot \cos \nu \cdot \cos A}} \\ &: \frac{\sin B}{\sqrt{\sin^2 B - \cos^2 \nu - \cos^2 \lambda + 2 \cos \nu \cdot \cos \lambda \cdot \cos B}} \\ &: \frac{\sin C}{\sqrt{\sin^2 C - \cos^2 \lambda - \cos^2 \mu + 2 \cos \lambda \cdot \cos \mu \cdot \cos C}}. \end{aligned}$$

Let a sphere cut the lines  $OA$ ,  $OB$ ,  $OC$  and the normal through  $O$  to  $m$  in  $A$ ,  $B$ ,  $C$ ,  $M$  (Fig. 12).

\* The angles  $\lambda$ ,  $\mu$ ,  $\nu$ ,  $A$ ,  $B$ ,  $C$  are the external angles of the tetrahedron formed by  $OBC$ ,  $OCA$ ,  $OAB$ , and  $m$ ; i. e. the angles actually measured by the goniometer.

The point  $M$  is called the *pole* of the face  $m$ .

The angles  $CAB, ABC, BCA$  of the spherical triangle  $ABC$  are the supplements of the angles  $A, B, C$ , which the planes  $OBC, OCA, OAB$  make with one another; hence the angles between the crystallographic axes are found from the formulae

$$\cos BOC = \frac{\cos B \cdot \cos C - \cos A}{\sin B \cdot \sin C}, \text{ \&c.}$$

The angles  $MA, MB, MC$  ( $A'OM', B'OM', C'OM'$  of Fig. 10) which determine the axial ratios are not independent. For since the sum of the angles  $BMC, CMA, AMB$  is  $2\pi$

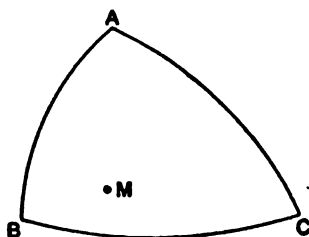


Fig. 12.

$$1 + 2 \cos BMC \cdot \cos CMA \cdot \cos AMB - \cos^2 BMC - \cos^2 CMA - \cos^2 AMB = 0.$$

$$\text{But } \cos BMC = \frac{\cos BC - \cos MB \cdot \cos MC}{\sin MB \cdot \sin MC}, \text{ \&c.,}$$

and  $\therefore$  substituting for  $\cos BMC, \cos CMA, \cos AMB$ , and reducing, we have

$$1 + 2 \cos BC \cdot \cos CA \cdot \cos AB - \cos^2 BC - \cos^2 CA - \cos^2 AB \\ = 2 \Sigma \{ \cos^2 MA \cdot \sin^2 BC + 2 \cos MB \cdot \cos MC (\cos CA \cdot \cos AB - \cos BC) \}.$$

Similarly the angles  $\lambda, \mu, \nu$  are not independent, for  $\pi - \lambda, \pi - \mu, \pi - \nu$  are the distances of  $M$  from the vertices of the polar triangle of  $ABC$  (whose sides are  $A, B, C$ ); and therefore

$$1 + 2 \cos A \cdot \cos B \cdot \cos C - \cos^2 A - \cos^2 B - \cos^2 C \\ = \Sigma \{ \cos^2 \lambda \cdot \sin^2 A + 2 \cos \mu \cdot \cos \nu (\cos B \cdot \cos C - \cos A) \},$$

$$\text{or } \Sigma \sin^2 \lambda \cdot \sin^2 A = 2 \{ 1 - \cos A \cdot \cos B \cdot \cos C \\ + \Sigma \cos \mu \cdot \cos \nu (\cos B \cdot \cos C - \cos A) \}^*.$$

This formula is a check on the accuracy of our goniometric measurement of the angles  $A, B, C, \lambda, \mu, \nu$ .

\* This equation gives the relation between the angles which any four planes make with one another. It may be put in the form

$$\begin{vmatrix} 1 & \cos C & \cos B & \cos A \\ \cos C & 1 & \cos A & \cos B \\ \cos B & \cos A & 1 & \cos C \\ \cos A & \cos B & \cos C & 1 \end{vmatrix} = 0.$$

See note II at the end of chap. vii.

§ 10. Let  $m_1$  be any other face of the crystal making angles  $\lambda_1, \mu_1, \nu_1$ , with  $OBC, OCA, OAB$ ; and let this face meet  $OA, OB, OC$  in  $A_1, B_1, C_1$ . Then if  $h, k, l$  are the indices of  $m_1$

$$\begin{aligned} h:k:l &= \frac{a}{OA_1} : \frac{b}{OB_1} : \frac{c}{OC_1} \\ &= \sqrt{\frac{\sin^2 A - \cos^2 \mu_1 - \cos^2 \nu_1 + 2 \cos \mu_1 \cdot \cos \nu_1 \cdot \cos A}{\sin^2 A - \cos^2 \mu - \cos^2 \nu + 2 \cos \mu \cdot \cos \nu \cdot \cos A}} \\ &: \sqrt{\frac{\sin^2 B - \cos^2 \nu_1 - \cos^2 \lambda_1 + 2 \cos \nu_1 \cdot \cos \lambda_1 \cdot \cos B}{\sin^2 B - \cos^2 \nu - \cos^2 \lambda + 2 \cos \nu \cdot \cos \lambda \cdot \cos B}} \\ &: \sqrt{\frac{\sin^2 C - \cos^2 \lambda_1 - \cos^2 \mu_1 + 2 \cos \lambda_1 \cdot \cos \mu_1 \cdot \cos C}{\sin^2 C - \cos^2 \lambda - \cos^2 \mu + 2 \cos \lambda \cdot \cos \mu \cdot \cos C}}. \end{aligned}$$

This gives us the indices of the face in terms of measured angles.

In practice it is not necessary to use this formula to obtain the indices of every face.

For instance, if five faces  $n_1, n_2, n_3, n_4, n_5$  are such that  $n_1, n_3, n_5$  lie in one zone, and  $n_2, n_3, n_4$  lie in another\*, we can find the indices of  $n_3$  when we have found the indices of  $n_1, n_2, n_4, n_5$ ; for from these latter we can obtain the indices of the two zonal axes, and from them we obtain the indices of  $n_3$  (p. 12).

Again, if four faces lie in a zone and we know the angles between all four faces, and the indices of three of the faces, we obtain the indices of the fourth face by the formula of page 15 (§ 7).

The assemblage of the poles (p. 19) of all the faces of a crystal, or, which is often more convenient, the stereographic projection of this assemblage, may be taken as the geometrical representation of the crystal.

There is no difficulty in drawing this projection for any crystal whose angles have been measured, with the help of chap. i, §§ 4, 5 (p. 6).

The poles of all faces in one zone lie on a great circle of the sphere, and their projections therefore lie on a circle

\* Our goniometric observations show us at once which faces lie in a zone; for if the crystal is adjusted so that a zonal axis is parallel to the axis of the instrument and is rotated about this axis, images of the slit can be seen in the telescope by reflexion at faces which belong to the zone, and at such faces only.

## SYMMETRICAL AND ASYMMETRICAL CRYSTALS 21

which passes through the ends of a diameter of the circle  $s$  (cf. Fig. 7, p. 6).

It is outside the purpose of this treatise to pursue this subject further. We have proved all the theorems necessary for the deduction of the geometrical properties of a crystal from goniometric measurements; for the details of the application of these theorems we must refer the reader to treatises on practical crystallography.

NOTE.—The formulæ of §§ 9 and 10 can be put in a form suitable for logarithmic calculation by the substitutions

$$\begin{aligned} \sin^2 A - \cos^2 \mu - \cos^2 \nu + 2 \cos \mu \cdot \cos \nu \cdot \cos A \\ = 4 \sin \frac{1}{2}(A + \mu + \nu) \cdot \sin \frac{1}{2}(-A + \mu + \nu) \\ \quad \cdot \sin \frac{1}{2}(A - \mu + \nu) \cdot \sin \frac{1}{2}(A + \mu - \nu). \end{aligned} \quad \text{etc}$$

## CHAPTER III

## SYMMETRY.

X § 1. A figure \* is said to be *translated* when each point of it is moved through the same distance in the same direction.

This motion (a *translation*) can be represented by a straight line whose length and direction (but not position) are given.

I A figure is said to be *rotated* about a straight line (called the 'axis of rotation') when the figure moves as if rigidly connected with the straight line, every point of which is fixed in space.

The angle between the final and initial positions of any plane parallel to the axis and rigidly connected with the figure is called the angle of rotation.

Each point of the figure evidently describes a circle about some point of the axis as centre.

This motion (a *rotation*) can be represented by a finite straight line (whose length gives the angle of rotation), which lies along the axis, but whose position in the axis is not fixed.

X § 2. Two figures are said to be *congruent* to one another when one can be brought to coincide with the other by a movement which does not alter the relative position of its own parts †.

Two figures which are congruent to the same figure are evidently congruent to one another.

A figure  $U$  can be brought into coincidence with any congruent figure  $U'$  by a translation followed by a rotation.

For let the point  $O$  of  $U$  correspond to the point  $O'$  of  $U'$ , and the lines  $OA, OB$  of  $U$  to the lines  $OA', OB'$  of  $U'$  (so that when  $U$  coincides with  $U'$ ,  $O$  coincides with  $O'$ ,  $OA$  with  $OA'$ ,  $OB$  with  $OB'$ ), and let  $U$  be translated through a distance  $OO'$  in a direction parallel to  $OO'$  ‡.

\* In future we shall mean by 'figure' any collection of points, lines, and planes.

† That is, a movement in which the figure behaves as a rigid body.

‡ Such a movement is represented in magnitude and direction by the straight line  $OO'$ , and is called 'the translation  $OO'$ '.

Then  $O$  is brought into coincidence with the corresponding point  $O'$ ; suppose that  $OA, OB$  are brought into the positions  $OA_1$  and  $OB_1$  respectively.

Let a sphere whose centre is  $O (= O')$  cut  $OA_1, OB_1$  in  $A_1, B_1$ , and the corresponding lines of  $U'$  in  $A'$  and  $B'$ ; then evidently the points  $A_1, B_1$  in  $U$  correspond to  $A', B'$  in  $U'$ .

Bisect the great circles  $A_1A', B_1B'$  at right angles by great circles meeting at  $I$  (Fig. 13).

Then obviously

$IA_1 = IA'; IB_1 = IB'$ ;  
also  $A_1B_1 = A'B'$ , for  
 $A_1, B_1$  correspond to  
 $A', B'$ .

Therefore the angle  
 $A_1IB_1 =$  the angle  
 $A'IB'$ , and hence the  
angle  $A_1IA' =$  the angle  
 $B_1IB'$ .

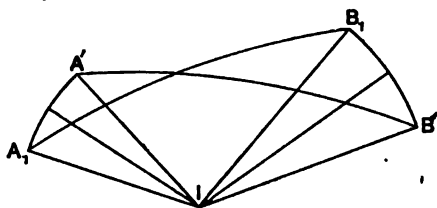


Fig. 13.

Therefore  $A_1$  is brought to  $A'^*$ , and  $B_1$  to  $B'$  by a rotation about  $OI$  through an angle  $A_1IA' (= B_1IB')$ .

Now a figure is completely fixed in position when the positions of three of its points are known.

Hence if three points of the figure  $U$  coincide with the corresponding points of a congruent figure  $U'$ ,  $U$  will completely coincide with  $U'$ .

But  $O$  is brought to  $O'$ ,  $A$  to  $A'$ , and  $B$  to  $B'$  by the translation  $OO'$  followed by a rotation about  $OI$ ;  $\therefore U$  is brought to coincide with  $U'$  by the translation  $OO'$  followed by a rotation about  $OI$ .

*A figure can be brought into coincidence with any congruent figure by a rotation followed by a translation.*

For since  $U$  is brought into coincidence with  $U'$  by a translation followed by a rotation;  $\therefore U'$  is brought into coincidence with  $U$  by a rotation followed by a translation.

§ 3. *A figure is rotated about an axis  $OA$  through an angle  $\alpha$ , and then about an axis  $OB$  through an angle  $\beta$ ; to find the resulting movement.*

By § 2, since the point  $O$  of the figure is fixed, therefore the resulting movement is a rotation about an axis through  $O$ .

We shall consider angles positive if they are measured in the clockwise direction.

Let  $OA, OB$  meet a sphere whose centre is  $O$  in  $A$  and  $B$ .

\* i. e. ' $A_1$  is brought into coincidence with  $A'$ '

Make the spherical angle  $CAB = -\frac{\alpha}{2}$  and the spherical angle  $ABC = -\frac{\beta}{2}$ .

Produce the great circle  $BC$  to  $W$ , draw the great circle  $AWE$  perpendicular to  $BCW$ , and take the point  $E$  such that  $AW = WE$  (Fig. 14).

Let  $C'AB = \frac{\alpha}{2}$  and  $ABC' = \frac{\beta}{2}$ .

Then  $C$  is brought to  $C'$  by the rotation about  $OA$  through the angle  $\alpha$ ; and  $C'$  is brought to  $C$  by the rotation about  $OB$  through an angle  $\beta$ .

Again  $A$  is left unmoved by the rotation about  $OA$  and is brought to  $E$  by the rotation about  $OB$ .

Hence  $C$  is left unmoved by the resultant of the rotations first about  $OA$  and then about  $OB$ , while  $A$  is brought to  $E$ .

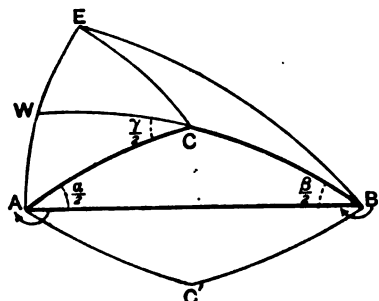


Fig. 14.

Therefore the resultant motion (which is some rotation about an axis through  $O$ ) must be a rotation about  $OC$  through an angle  $ECA = 2WCA$  in the positive (clockwise) direction.

The above is called Euler's or Rodrigues' construction.

From the spherical triangle  $ABC$  we have, if  $\phi$  is the angle between  $OA$  and  $OB$ ,

$$\cos \frac{\alpha}{2} \cdot \cos \frac{\beta}{2} - \sin \frac{\alpha}{2} \cdot \sin \frac{\beta}{2} \cdot \cos \phi = \cos ACW = \cos \frac{\gamma}{2},$$

if  $\gamma$  is the angle of rotation about  $OC$ ; and

$$\therefore \frac{1}{2} \{ (\cos \alpha + \cos \beta - 1) \sin^2 \phi + \cos \alpha \cdot \cos \beta \cdot (1 + \cos^2 \phi) - 2 \sin \alpha \cdot \sin \beta \cdot \cos \phi \} = \cos \gamma.$$

As a particular case we have  $\phi = 0$ ; the axes  $OA, OB$  are then parallel and  $O$  is at infinity. The same geometrical construction holds for the position of the axis of the resultant rotation. This axis is now parallel to  $OA$  and  $OB$ , and  $\gamma = \alpha + \beta$ .

The resultant of a rotation first about  $OB$  and then about  $OA$  is easily seen to be a rotation through the same angle  $\gamma$  in the same direction about the axis  $OC$ .

As another particular case take  $\alpha = \beta = \pi$ .

Then  $OC$  is perpendicular to the plane  $OAB$  and

$$\cos \frac{\gamma}{2} = -\cos \phi. \quad \therefore \frac{\gamma}{2} = \pi - \phi.$$

If  $\phi = \frac{\pi}{2}$ ;  $\alpha = \beta = \gamma = \pi$ .

§ 4. If from any point  $O$  of a figure  $U$ , a perpendicular  $ON$  is drawn to a fixed plane  $q$ , and produced to  $O'$  such that  $OO'$  is bisected at  $N$ ; then  $U'$ , the locus of  $O'$ , is called the reflection of  $U$  in the plane  $q$ .

If  $U'$  is the reflexion of  $U$ , evidently  $U$  is the reflexion of  $U'$  in the same plane  $q$ .

If  $C$  is a fixed point, and  $O$  any point of a figure  $U$ ; and if  $OC$  is produced to  $O''$  so that  $OO''$  is bisected at  $C$ ; then  $U''$ , the locus of  $O''$ , is called the inverse of  $U$  about the point  $C$ \*

If  $U''$  is the inverse of  $U$ , evidently  $U$  is the inverse of  $U''$  about the same point.

If the point  $C$  lies in the plane  $q$ ; then  $U''$  can be brought into coincidence with  $U'$  by a rotation through  $\pi$  about an axis through  $C$  perpendicular to  $q$ .

For this axis being perpendicular to  $q$ , is parallel to  $OO'$ , and lies therefore in the plane  $O''OO'$ ; let it meet  $O'O''$  in  $K$  (Fig. 15).

Then  $ON = NO'$  and  $OC = CO''$ , therefore  $O'O''$  is parallel to  $CN$ ; and therefore, since  $CK$  is parallel to  $OO'$ , and  $O''C = CO$ , the triangles  $O''CK$ ,  $CON$  are equal in all respects.

$\therefore O''K = CN = KO'$  (for  $CNO'K$  is a parallelogram), and the angle  $CKO'' = ONC =$  a right angle.

Hence  $O''$  is brought to  $O'$  by a rotation through  $\pi$  about  $CK$ ; and this being true whatever point  $O$  of  $U$  we start with, must be true for all the points of  $U'$  and  $U''$ , and therefore for the whole figures. Evidently  $U'$  is brought to  $U''$  † by the same rotation.

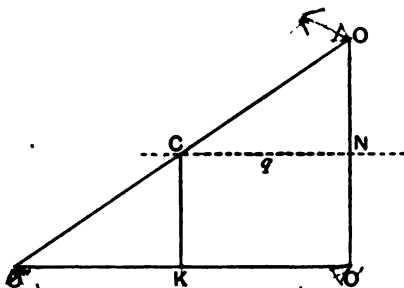


Fig. 15.

\* Distinguish from the 'inverse with respect to a point'; cf. chap. i, § 2, p. 1.

† i.e. 'the figure  $U$  is brought into coincidence with  $U''$ ' and so in future.

The theorem is also readily proved by consideration of Fig. 5 (p. 4); it may be stated in the form :

If  $U''$  is the inverse of  $U$  about a point  $C$ ; and if  $U''$  be brought to  $U'$  by rotation through two right angles about any axis through  $C$ ; then  $U'$  is the reflexion of  $U$  in a plane through  $C$  at right angles to the axis.

§ 5. If  $U'$  is the reflexion of a figure  $U$  in a plane  $q_1$  and  $U''$  is the reflexion of  $U$  in a plane  $q_2$ ; then  $U'$  can be brought to coincide with  $U''$  by a rotation through an angle  $2\alpha$  about the intersection of  $q_1$  and  $q_2$ , where  $\alpha$  is the angle between the planes  $q_1$  and  $q_2$ .

For let  $O'$  be the reflexion in  $q_1$  of any point  $O$  of  $U$ ; and  $O''$  the reflexion of  $O$  in  $q_2$  (Fig. 16).

Let  $OO'$  meet  $q_1$  in  $M$ , and let  $OO''$  meet  $q_2$  in  $N$ .

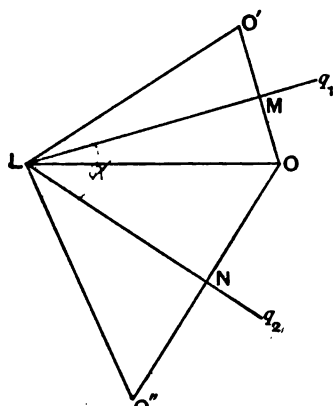


Fig. 16.

Then the plane  $O'O''$  is perpendicular to  $q_1$  and  $q_2$ , and therefore cuts their line of intersection at right angles (in  $L$  say).

Then because  $LM$  bisects  $OO'$  at right angles, therefore  $OL = O'L$  and the angle  $OLO' = 2 \cdot OLM$ .

Similarly  $OL = O''L$  and

$$O''LO = 2 \cdot NLO.$$

Therefore  $O'L = O''L$  and  $O''LO + OLO' = 2(NLO + OLM)$ , or  $O''LO' = 2 \cdot NLM = 2\alpha$ .

Hence  $O'$  is brought to  $O''$  by a rotation through  $2\alpha$  about the intersection of  $q_1$  and  $q_2$ ; and

this being true whatever point  $O$  of  $U$  we started from, it is true for the whole figures  $U, U', U''$ .

Corollary 1. Two figures which are reflexions of the same figure are congruent to one another.

Corollary 2. If  $U'$  is the reflexion of any figure  $U$  in a plane  $q$ , and if  $U'$  is brought to coincidence with  $U''$  by a rotation through an angle  $2\alpha$  about an axis  $l$  lying in  $q$ , then  $U''$  is the reflexion of  $U$  in a plane through  $l$  which makes an angle  $\alpha$  with  $q$ .

§ 6. Two figures are said to be *enantiomorphous* when one is the reflexion of the other in any plane, or when one can be brought by a movement which leaves the relative positions of

its parts unaltered into such a position that it is the reflexion of the other in a plane.

A figure and its inverse about a point are enantiomorphous by § 4.

Two figures  $U'$  and  $U''$  which are enantiomorphous to the same figure  $U$ , are congruent to one another. For by definition  $U'$  is congruent to some figure  $U_1'$  which is the reflexion of  $U$  in some plane, and  $U''$  is congruent to some figure  $U_1''$  which is the reflexion of  $U$  in some other plane. Now by the previous section  $U_1'$  and  $U_1''$  are congruent, hence  $U'$ ,  $U_1''$  being both congruent to  $U_1'$  are congruent to one another; and  $\therefore U'$  and  $U''$  being both congruent to  $U_1''$  are congruent to one another.

*A figure  $U$  can be brought into such a position that it is the reflexion of any enantiomorphous figure  $U'$  in a plane, by a rotation about an axis perpendicular to the plane.*

For let  $O, O'$  be corresponding points in  $U$  and  $U'$ . Bisect  $OO'$  in  $E$ ; let  $U''$  be the inverse of  $U'$  about  $E$ ; then  $U$  and  $U''$  being both enantiomorphous to  $U'$  are congruent to one another. Also the point  $O$  of  $U''$  corresponds to  $O'$  of  $U'$  and therefore to  $O$  of  $U$ ; i.e.  $U$  and  $U''$  have two corresponding points (one in each figure) identical. Hence  $U$  can be brought to coincide with  $U''$  by a rotation about some axis  $l$  passing through  $O$ . Let  $m$  be the line through  $E$  parallel to  $l$ . Then by the end of § 4, if  $U''$  be brought to  $U'''$  by a rotation through  $\pi$  about the axis  $m$ ,  $U'''$  is the reflexion of  $U'$  in a plane through  $E$  perpendicular to  $l$  or  $m$ .

Now  $U$  is brought to  $U'''$  by two consecutive rotations about the parallel axes  $l, m$ ; and these rotations are by § 3 equivalent to some single rotation about an axis  $n$  parallel to  $l^*$ . But  $U'''$  is the reflexion of  $U'$  in a plane perpendicular to  $l$  or  $n$ ; hence  $U$  can be brought by a rotation about an axis  $n$  into coincidence with the reflexion of  $U'$  in a plane perpendicular to  $n$ .

We may put the results of this section in the following form:

*A figure may be brought to coincidence with any enantiomorphous figure by a rotation about an axis followed by reflexion in a plane perpendicular to that axis\*.*

This combination of a rotation about an axis through an

\* If the angle of rotation about  $l$  is  $\pi$ ,  $n$  is at infinity and the angle of rotation about  $n$  is infinitesimal; that is,  $U$  is brought to  $U'''$  by a translation perpendicular to  $l$ , or  $U$  is brought to coincidence with  $U'$  by a translation followed by reflexion in a plane parallel to the direction of the translation.

angle  $\omega$  followed by a reflexion in a plane perpendicular to that axis, is called a *rotatory-reflexion*\* of angle  $\omega$ .

Since  $U$  is brought to  $U'$  by a rotation followed by a reflexion,  $\therefore U'$  is brought to  $U$  by a reflexion followed by a rotation; hence:

*A figure may be brought to coincidence with any enantiomorphous figure by a reflexion in a plane followed by a rotation about an axis perpendicular to that plane.*

Any two enantiomorphous figures have two corresponding points (one in each) common; for the point where the axis  $n$  meets the perpendicular plane through  $E$  corresponds to itself in the two figures †.

§ 7. A translation, rotation, reflexion or the result of combining any of these is called an *operation* ‡. An operation acting on a figure must bring it into coincidence, either with a congruent figure, when the operation is said to be *of the first sort*; or with an enantiomorphous figure, when the operation is said to be *of the second sort*. An operation of the first or second sort is also called a *movement*. It should be noticed that the movements with which we are concerned, are those which do not alter the distance between any two given points.

An operation may conveniently be represented by a symbol; thus a rotation about an axis  $a$  through an angle  $\alpha$  may be represented by  $A$  or  $A(\alpha)$ ; a reflexion by  $S$ , an inversion about a point by  $I$ , &c.

If two or more operations are written after one another it means that the operations are applied to the figure under consideration in order from left to right.

Thus  $A.B.C$  means that first the operation  $A$  acts, then the operation  $B$ , and finally the operation  $C$ ; the resulting movement is called the 'product' or 'resultant' of  $A$ ,  $B$ , and  $C$ .

If in § 3 (p. 24)  $A(\alpha)$  is the rotation about  $OA$ ,  $B(\beta)$  that about  $OB$ ,  $C(\gamma)$  that about  $OC$ ; we may write the results there arrived at in the form:

$$A(\alpha).B(\beta) = C(\gamma), \text{ or } A.B = C; B(\beta).A(\alpha) = C'(\gamma), \\ \text{or } B.A = C'$$

(cf. Fig. 14); where the sign of equality (=) denotes identity of effect on *any* figure.

\* German—Drehspiegelung.

† In the case mentioned in the footnote on p. 27 this point is at infinity.

‡ This word is usually used in a far wider sense; but translations, rotations, reflexions, and combinations of these are the only kinds of 'operation' considered in this book.

If  $A, B$  are two operations such that  $A.B = B.A$ ,  $A$  and  $B$  are said to be *permutable*; from what has just been said we see that two rotations about intersecting axes are not in general permutable. If it is desired to specify the figure on which any operation or series of operations acts, some symbol denoting the figure is placed in square brackets before the symbol or symbols denoting the operation or series of operations.

If a figure remains fixed we say that *the identical operation* acts on it; the identical operation is denoted by the symbol 1. We have evidently  $1.A = A.1 = A$ , where  $A$  is any operation. The operation  $A.A.A \dots$  to  $n$  factors is written  $A^n$ .

The reverse operation to  $A$  is written  $A^{-1}$ , so that

$$A.A^{-1} = A^{-1}.A = 1.$$

$A^{-1}.A^{-1}.A^{-1} \dots$  to  $n$  factors is written  $A^{-n}$ .

If  $A, B, C$  are any operations, and  $A.B = P, B.C = Q$ , then  $P.C = A.Q$ .

For let  $U$  be any figure, and let

$$[U]A = [U']1, [U']B = [U'']1, [U'']C = [U''']1;$$

that is, let  $A$  bring  $U$  to  $U'$ ,  $B$  bring  $U'$  to  $U''$ , and  $C$  bring  $U''$  to  $U'''$ .

Then  $[U]P = [U]A.B = [U']B = [U'']1$ , and therefore  $[U]P.C = [U''']1$ .

Similarly  $[U']Q = [U']B.C = [U'']C = [U''']1$ , and therefore  $[U]A.Q = [U''']1$ .

Hence since  $U$  is any figure  $P.C = A.Q$ .

This result may be written  $(A.B).C = A.(B.C)$ ; it may readily be extended, and we see that in general operations of the kind we are considering obey the associative law though not the commutative.

We deduce from this that

$$A^{-2}.A^2 = A^{-1}.(A^{-1}.A).A = A^{-1}.A = 1.$$

$$A^{-3}.A^3 = A^{-1}.(A^{-2}.A^2).A = A^{-1}.A = 1,$$

and in general  $A^{-n}.A^n = 1$ .

Similarly  $A^n.A^{-n} = 1$ .

It readily follows that  $A^p.A^q = A^{p+q}$ , where  $p$  and  $q$  are any positive or negative integers; if by  $A^0$  we mean the identical operation.

If  $C$  and  $D$  are equivalent operations so that  $C = D$ ; then  $C.A = D.A$  and  $A.C = A.D$ , where  $A$  is any operation.

For let  $U$  be any figure, and let  $[U]C = [U']1, [U']A = [U'']1$ ,

Then because  $C = D$ , therefore  $[U] D = [U] 1$ .

Hence  $[U] C.A = [U] A = [U''] 1$ , and  $[U] D.A = [U] A = [U''] 1$ ; and therefore  $C.A$  and  $D.A$  both bring  $U$  to  $U''$ , so that  $C.A = D.A$ .

We may prove  $A.C = A.D$  in a similar way.

Hence we may multiply both sides of an equation of operations either on the right or on the left by the same operation.

Similarly, if  $C.A = D.A$ .

Then  $C.A.A^{-1} = D.A.A^{-1}$  by what we have just proved, and  $\therefore C = D$ .

Again, if  $A.C = A.D$  then  $A^{-1}.A.C = A^{-1}.A.D$  and  $\therefore C = D$ .

Hence we may divide both sides of an equation of operations either on the right or on the left by the same operation.

§ 8. *The product of two operations is an operation of the first or second sort according as the operations are of the same or of different sorts.*

Let  $L, M$  be any two operations, and let  $L$  bring any figure  $U$  to  $U'$ , and let  $M$  bring  $U'$  to  $U''$ .

First suppose  $L$  and  $M$  of the same sort.

Then  $U$  and  $U''$  are either both congruent to  $U'$  or both enantiomorphous to  $U'$ ; and hence  $U$  and  $U''$  are congruent.

But  $L.M$  brings  $U$  to  $U''$ ; therefore  $L.M$  is an operation of the first sort.

Next let  $L, M$  be two operations of different sorts; in this case either  $U'$  is congruent to  $U$  and not to  $U''$ , or to  $U''$  and not to  $U$ . Hence  $U$  and  $U''$  are enantiomorphous; and therefore, since  $L.M$  brings  $U$  to  $U''$ ,  $L.M$  is an operation of the second sort.

We may readily extend the above and obtain the theorem:—

*The product of any number of operations is an operation of the first or second sort, according as the number of operations of the second sort in the product is even or odd.*

§ 9. If  $A(a), A(\beta), A(\gamma), \dots$  are rotations through  $a, \beta, \gamma, \dots$  about the same axis, we have evidently  $A(a).A(\beta).A(\gamma) \dots = A(a + \beta + \gamma + \dots)$ .

In particular  $A^*(a)^* = A(na)$ .

Again, since  $A(a).A(-a) = A(o) = 1$ .

$\therefore A(-a) = A^{-1}(a)$ .

\* We shall denote  $\{A(a)\}^*$  by  $A^*(a)$  in future.

§ 10. If  $A(a)$  be a rotation through  $a$  about an axis  $a$ , and  $S$  a reflexion in a plane perpendicular to  $a$ , then  $A(a)$  and  $S$  are permutable operations.

The point  $O$  in which the axis and plane meet is evidently left unmoved by either operation.

Let  $P'$  be any point; on a plane perpendicular to  $a$  form the stereographic projection of the sphere whose centre is  $O$  and radius  $OP'$ .

Then (using the notation of p. 4),  $P(+)$ , the projection of  $P'$ , is brought by  $A(a)$  to  $P_1(+)^*$ , (Fig. 17), where  $P_1CP = a$ , and  $CP = CP_1$ ; and  $P_1(+)$  is brought to  $P_1(o)$  by  $S$ .

Therefore  $A(a).S$  brings  $P(+)$  to  $P_1(o)$ .

Again  $S$  brings  $P(+)$  to  $P(o)$ , and  $A(a)$  brings  $P(o)$  to  $P_1(o)$ ; therefore  $S.A(a)$  brings  $P(+)$  to  $P_1(o)$ .

Hence, since  $P'$  is any point,  $A(a).S = S.A(a)$ . As a particular case  $A(\pi).S = S.A(\pi) = I$ , the inversion about  $O$ .

$A(a).S = S.A(a)$  is the general operation of the second sort (rotatory-reflexion).

§ 11. Let  $U$  be any figure, and  $A'(a) = A(a).S$  be this general operation of the second sort; we shall consider the properties of powers of  $A'$ .

First let  $a = o$ .

If  $U$  is any figure and  $[U]S = [U']1$ ; then we know that  $[U']S = [U]1^\dagger$ . Therefore  $[U]S^2 = [U']S = [U]1$ , and hence  $S^2 = 1$ .

It follows that  $S^{2m} = (S^2)^m = 1^m = 1$ , and that

$$S^{2m+1} = 1.S = S;$$

where  $m$  is any integer.

Next let  $a = \pi$ . Then  $I^2 = A(\pi).S.S.A(\pi) = A(\pi).A(\pi) = A(2\pi) = 1$ .

It follows that  $I^{2m} = 1$ , and that  $I^{2m+1} = I$ .

Now suppose that  $a$  is not equal to  $o$  or a multiple of  $\pi$ .

Then  $A'^2(a) = A.S.A.S = A.A.S.S = A^2(a) = A(2a)$ ; and  $A'^3(a) = A(2a).A.S = A(3a).S$ .

Similarly  $A'^4(a) = A(4a)$ ;  $A'^5(a) = A(5a).S$ , &c.

\* i. e. the point of which  $P(+)$  is the projection is brought into coincidence with the point of which  $P_1(+)$  is the projection.

† That is, 'if  $U'$  is the reflexion of  $U$  in any plane,  $U$  is the reflexion of  $U'$  in the same plane' (p. 25).

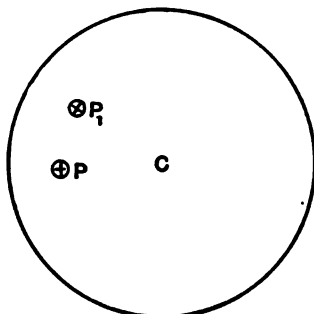


Fig. 17.

Hence the operations

$$\left. \begin{array}{l} 1, A'^2, A'^4, A'^6, \dots \\ A', A'^3, A'^5, A'^7, \dots \end{array} \right\}$$

are equivalent to

$$1, A(a).S, A(2a).S, A(4a).S, A(6a).S, \dots \}.$$

§ 12. If  $U$ ,  $U'$  be any two congruent or enantiomorphous figures then  $U$  can be brought into coincidence with  $U'$  by a rotation, a translation, a combination of rotation and translation, or a rotatory-reflexion.

Now if  $U'$  is identical with  $U$ ,  $U$  is brought into coincidence with itself by some operation of the kind just described. In that case  $U$  is said to have *symmetry*, and the operation which brings  $U$  to self-coincidence is called a *symmetry-operation* of  $U$ . It may be possible to bring  $U$  to self-coincidence\* in several different ways, that is, by means of several different symmetry-operations; but in every case the operation must be equivalent to a rotation, a translation, a combination of rotation and translation, or a rotatory-reflexion; hence:

*The only kinds of symmetry a figure can possess are those due to a rotation, a translation, a combination of rotation and translation, or a rotatory-reflexion.*

In this statement a simple reflexion, a translation followed by a reflexion in a parallel plane, and inversion about a point are included as special cases of a rotatory-reflexion.

If a figure  $U$  is brought to self-coincidence by a rotation about any line, that line is called a *rotation-axis*, 'axis of symmetry of the first sort,' or 'axis of symmetry' of  $U$ .

If a figure  $U$  coincides with its own reflexion in a plane, that plane is called a *symmetry-plane* or 'plane of symmetry' of  $U$ .

If a figure  $U$  coincides with its own inverse about a point, that point is called a *centre of symmetry* or 'centre of inversion' of  $U$ .

\* When we say that a figure  $U$  is brought to self-coincidence, we do not mean that each point, line, or plane of  $U$  is brought to coincide with itself; but that each point, line, or plane of  $U$  is brought into the position previously occupied either by itself or by some other point, line, or plane of  $U$ .

For example: if the cube  $AA_1A'A_1'A''A_1''A_1'''A_1''''$  (Fig. 126) is rotated through  $\frac{2\pi}{3}$  about the diagonal  $AA_1$ , the line  $AA_1$  does not alter its position, while the line  $A'A_1'$  is brought into the position previously occupied by  $A''A_1''$ , the line  $A_1A'$  into the position previously occupied by  $A_1A''$ , &c.; but the cube as a whole is brought into self-coincidence by the rotation.

If a figure  $U$  is brought to self-coincidence by a rotatory-reflexion about an axis, that axis is called a *symmetry-axis of the second sort* or 'symmetry-axis of rotatory-reflexion' of  $U$ .

Such axes, planes, &c. are called collectively *symmetry-elements* or 'elements of symmetry' of  $U$ .

The identical operation is by definition a symmetry-operation of any figure.

§ 13. Let  $A, B, C, \dots, L$  be symmetry-operations of  $U$ , so that  $[U]A = [U]B = [U]C = \dots = [U]L = [U]1$ . Then  $[U]A.B.C.\dots.L = [U]B.C.\dots.L = [U]C.\dots.L = \dots = [U]L = [U]1$ .

Hence if  $A, B, C, \dots, L$  are symmetry-operations of any figure,  $A.B.C.\dots.L$  is also a symmetry-operation.

Again, if  $H$  is a symmetry-operation of  $U$ ,

$$[U]H^{-1} = [U]H.H^{-1} = [U]1; \text{ hence:—}$$

If  $H$  is a symmetry-operation of any figure so is  $H^{-1}$ .

Combining the two previous theorems we have:

If  $A, B, C, \dots, L$  are symmetry-operations of any figure,  $A^p.B^q.C^r.\dots.L^z$  is also a symmetry-operation; where  $p, q, r, \dots, z$  are any positive or negative integers.

A particular case of this is the following theorem:

If  $A$  is a symmetry-operation of any figure so is  $A^n$ ; where  $n$  is any positive or negative integer.

For example; if the rotations  $A(\alpha), A(\beta), A(\gamma), \dots$  about the same axis are symmetry-operations of any figure; so are  $A(n\alpha), A(n\beta), A(n\gamma), \dots$  and in general  $A(pa + qb + rc + \dots)$ ; where  $n, p, q, r, \dots$  are any positive or negative integers.

§ 14. Let  $\alpha$  be the smallest positive angle through which a figure  $U$  must be turned about a rotation-axis  $a$  to bring it to self-coincidence; then, if  $\alpha$  is not infinitesimal,  $2\pi$  is a multiple of  $\alpha$ .

For if  $A(\alpha)$  represent a rotation through  $\alpha$  about  $a$ , then evidently  $A(\alpha)$  is identical with  $A(\alpha + 2m\pi)$  where  $m$  is a positive or negative integer.

Now if  $2\pi$  is not a multiple of  $\alpha$  let  $2\pi$  lie between  $r\alpha$  and  $(r+1)\alpha$ , where  $r$  is a positive integer; then  $\alpha > (r+1)\alpha - 2\pi > 0$ . But since  $A(\alpha)$  brings  $U$  into self-coincidence, therefore  $A^{r+1}(\alpha) = A((r+1)\alpha) = A((r+1)\alpha - 2\pi)$  does so. Hence

a rotation  $A(\overline{r+1}a - 2\pi)$  about  $a$  whose angle is positive and  $< a$  brings  $U$  to self-coincidence, contrary to hypothesis.

Let  $2\pi = na$ , or  $a = \frac{2\pi}{n}$ ; where  $n$  is a positive integer; then the axis is called a digonal, trigonal, tetragonal, pentagonal, hexagonal, ... or 2-al, 3-al, 4-al, 5-al, 6-al, ... axis of symmetry according as  $n = 2, 3, 4, 5, 6, \dots$ .

The only distinct symmetry-operations of  $U$  due to rotations about  $a$  are  $1, A, A^2, \dots, A^{n-1}$ ; or  $1, A(a), A(2a), \dots, A(\overline{n-1}a)$ .

For evidently  $A^{(rn+s)}$  is equivalent to  $A^s$  and  $A^{(n-s)}$  to  $A^{n-s}$  ( $r$  being any integer, and  $s$  zero or a positive integer  $< n$ ); so that every power of  $A$  is equivalent to one or other of the series  $1, A, A^2, \dots, A^{n-1}$ .

Again if  $A(\omega)$  is a symmetry-operation of  $U$ ,  $\omega$  is a multiple of  $a$ . For otherwise it would be possible to find an integer  $p$  such that  $pa + \omega$  is a positive angle  $< a$  and  $> 0$ ; and in that case  $A(pa + \omega)$  would be a rotation through an angle  $< a$  and  $> 0$  bringing  $U$  to self-coincidence, contrary to hypothesis.

§ 15. Now let  $A' = A(a).S$  be a symmetry-operation of a figure  $U$  (using the notation of p. 31), where  $a$  is positive and  $\neq 0$  or  $\pi$ , and suppose that no rotatory-reflexion  $A(\gamma).S$  about the same axis and plane is a symmetry-operation of  $U$  if  $\gamma$  lies between  $0$  and  $a$  exclusive.

Let  $\beta$  be the smallest positive angle (zero excluded) for which the rotation  $A(\beta)$  is a symmetry-operation of  $U$ ; then since  $A'^2 = A(2a)$  is a symmetry-operation of  $U$ ,  $\beta \geq 2a$ .

Now since  $A(a).S, A(-2a), A(\beta), A(-\beta)$  are symmetry-operations of  $U$ , so are  $A(-\beta).A(a).S = A(a-\beta).S$ , and  $A(\beta).A(-2a).A(a).S = A(\beta-a).S$ .

Hence  $\beta \sim a$  is either  $0$  or else  $\leq a$ . But  $\beta \geq 2a$  and  $\neq 0$ , therefore  $\beta = a$  or  $2a$ .

Now  $2\pi$  is a multiple of  $\beta$  (p. 33), therefore  $2\pi$  is a multiple of  $a$ .

Hence if  $na = 2\pi$ ,  $n$  is a positive integer; the axis of the rotatory-reflexion is then called an  $n$ -al symmetry-axis of the second sort.

First suppose  $n$  odd.

In this case  $(A'^2)^n = A^n(2a) = 1$ ;

Hence evidently the only distinct powers of  $A'$  are

$$\left\{ \begin{array}{ccccccc} 1, & A(2a), & A(4a), & \dots, & A(\overline{2n-2}a) \\ A(a).S, & A(3a).S, & A(5a).S, & \dots, & A(\overline{2n-1}a) \end{array} \right\}$$

(cf. p. 32).

Since  $A(n+r)a = A(2\pi+ra) = A(ra)$ , these are equivalent to

$$\begin{aligned} & \{1, A(2a), A(4a), \dots, A(\overline{n-3}a), A(\overline{n-1}a), \\ & \{A(a).S, A(3a).S, A(5a).S, \dots, A(\overline{n-2}a).S, S, \\ & \quad A(a), A(3a), \dots, A(\overline{n-2}a)\} \\ & \quad A(2a).S, A(4a).S, \dots, A(\overline{n-1}a)\} \end{aligned}$$

or (rearranging) to

$$\begin{aligned} & \{1, A(a), A(2a), A(3a), \dots, A(\overline{n-1}a) \} \dots (i). \\ & \{S, A(a).S, A(2a).S, A(3a).S, \dots, A(\overline{n-1}a).S\} \dots (i). \end{aligned}$$

It is readily proved that these are the only operations, whether rotations or rotatory-reflexions, about the axis of  $A(a).S$ , which are symmetry-operations of  $U$ .

Next suppose  $n$  even.

In this case  $(A')^{\frac{n}{2}} = 1$ ; hence evidently the only distinct powers of  $A'$  are

$$\begin{aligned} & \{1, A(2a), A(4a), \dots, A(\overline{n-2}a)\} \\ & \{A(a).S, A(3a).S, A(5a).S, \dots, A(\overline{n-1}a).S\} \dots (ii). \end{aligned}$$

If  $n$  is a multiple of 4, we use this arrangement; if however  $n$  is a multiple of 2 but not of 4 we arrange the operations in the second line of (ii) in the order

$$\begin{aligned} & A\left(\frac{n}{2}a\right).S, A\left(\frac{n+4}{2}a\right).S, \dots, A(\overline{n-1}a).S, A(a).S, \\ & \quad A(3a).S, \dots, A\left(\frac{n-4}{2}a\right).S. \end{aligned}$$

If an inversion about the intersection of the axis and plane of the rotatory-reflexion be denoted by  $I = A(\pi).S$ , these are equivalent to  $I, A(2a).I, A(4a).I, \dots, A(\overline{n-2}a).I$ ,

$$\text{since } \frac{na}{2} = \pi, \text{ and } A(2\pi) = 1.$$

(ii) may then be written

$$\begin{aligned} & \{1, A(2a), A(4a), \dots, A(\overline{n-2}a)\} \\ & \{I, A(2a).I, A(4a).I, \dots, A(\overline{n-2}a).I\} \dots (iii). \end{aligned}$$

We may put the results of this section in the following form.

Let a line  $a$  meet a perpendicular plane  $\sigma$  in the point  $O$ .

Then if  $a$  is an  $n$ -al symmetry-axis of the second sort of any figure  $U$ , and  $\sigma$  is the corresponding plane,  $n$  being odd;  $a$  is an  $n$ -al rotation-axis of  $U$  and  $\sigma$  is a symmetry-plane.

Conversely if  $a$  is an  $n$ -al rotation-axis,  $n$  being odd, and  $\sigma$  is a symmetry-plane, then  $a$  is an  $n$ -al symmetry-axis of the second sort.

Again if  $a$  is an  $n$ -al symmetry-axis of the second sort and  $\sigma$  the corresponding plane,  $n$  being *even*, then  $a$  is a  $\frac{n}{2}$ -al rotation-axis; and if  $n$  is even but  $\frac{n}{2}$  odd,  $O$  is a centre of symmetry.

Conversely if  $a$  is an  $\frac{n}{2}$ -al rotation-axis,  $\frac{n}{2}$  being an odd integer, and  $O$  is a centre of symmetry, then  $a$  is an  $n$ -al symmetry-axis of the second sort.

§ 16. The normals from a fixed point  $O$  within a crystal to the crystal faces form a figure, which is characteristic of the substance. This figure is often found to possess symmetry. Similarly if radii vectores are drawn from  $O$  to represent any physical property of the crystal in different directions about  $O$ , the figure consisting of these vectors has also often elements of symmetry. The symmetry of all the figures formed by using different physical properties and the natural growth of faces is not always the same; the crystal is said to possess those elements of symmetry, and those only, which are common to *all* such figures.

It is found that usually the system of radii vectores representing any physical property has all the symmetry-elements possessed by the system of normals through a point to the faces. We shall therefore for the present (while we are dealing with crystal symmetry) mean by the word 'crystal' either (1) a collection of finite lines drawn through any point  $O$  perpendicular to the faces, only those lines perpendicular to faces whose physical properties—lustre, hardness, &c.—are the same being of equal length, or (2) a collection of planes parallel to the faces, those planes parallel to faces whose physical properties are the same being equidistant from  $O$ , or (3) the finite closed polyhedron formed by such planes.

It is indifferent whether we take the collection of lines, the collection of planes, or the polyhedron, as the 'crystal,' when we are only concerned with the symmetry; for any operation which brings one of them to self-coincidence obviously brings the other two also to self-coincidence. When, however, we are dealing with properties of crystal edges and faces the polyhedron is the most convenient representation, for its surfaces and edges are parallel to the actual faces and edges of the crystal as found in nature\*.

\* In any crystal found in nature faces whose physical properties are the same are not in general equidistant from any point inside the crystal, owing

The point  $O$  (which may be chosen at will) is called the *centre* of the crystal; it is evidently left unmoved by any operation which brings the 'crystal' to self-coincidence; hence:

*Any symmetry-operation of a crystal leaves its centre unmoved.*

It follows from this that the only symmetry-operations which a crystal can have are rotations about axes passing through its centre, reflexions in planes through its centre, rotatory-reflexions whose axes meet the corresponding planes at the centre, and an inversion about the centre.

§ 17. *Any symmetry-plane of a crystal is parallel to a possible\* crystal face, and a line perpendicular to it is parallel to a possible crystal edge.*

For let  $p, q$  be planes parallel to two possible faces, and let  $p', q'$  be their reflexions in the symmetry-plane ( $\sigma$ ). Then  $p', q'$  are also parallel to crystal faces, since  $\sigma$  is a symmetry-plane of the crystal. But the intersections of  $p, p'$  and of  $q, q'$  lie in  $\sigma$ ; hence  $\sigma$  contains lines parallel to two possible edges and is therefore parallel to a possible crystal face.

Again let  $u, v$  be parallels through  $O$  to any possible edges, and  $u', v'$  their reflexions in  $\sigma$ ; then the plane through  $u, u'$  and the plane through  $v, v'$  are parallel to possible faces and they meet in a perpendicular to  $\sigma$ , which is therefore parallel to a possible edge.

*Corollary.*  $\sigma$  contains an indefinite number of lines parallel to possible edges; for the intersection of  $\sigma$  with a plane parallel to any possible face is parallel to a possible edge.

§ 18. *An axis of symmetry is parallel to a possible crystal edge, and the plane perpendicular to it is parallel to a possible crystal face.*

Case (1).  $n$ -al axes of the first sort †;  $n$  even.

(a).  $n = 2$ . Let  $OB$  be the axis of symmetry and  $Oe, Of$  parallels to any two possible crystal edges. Let  $Oe', Of'$  be the positions to which  $Oe, Of$  would be brought by a rotation through  $\pi$  about  $OB$  (Fig. 18).

Then  $Oe', Of'$  are parallel to possible edges; and hence the planes  $Oee', Of'f'$  are parallel to possible faces; therefore their line of intersection  $OB$  is parallel to a possible edge.

to extraneous disturbances during the growth of the crystal; but we may assume that, had the growth been undisturbed, such faces would be all equidistant from some internal point.

\* Cf. end of § 5, chap. ii, p. 18.

† i. e. rotation-axes.

Again the planes  $Oef$ ,  $Oe'f'$  are parallel to possible faces and meet in a line perpendicular to  $OB$  which is parallel to a possible edge.

Similarly the planes  $Oef'$ ,  $Oe'f$  are parallel to possible faces and meet in a line perpendicular to  $OB$  which is parallel to a possible edge; hence the plane perpendicular to  $B$  contains two lines parallel to possible edges, and is therefore parallel to a possible face.

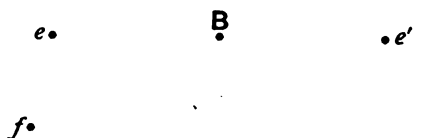


Fig. 18\*.

( $\beta$ ).  $n = 4$ . Let  $OB$  be the axis of symmetry, and let  $Oe_1$  be parallel to any possible edge. Let rotations about  $OB$  through  $\frac{\pi}{2}$  bring  $Oe_1$ , successively into the positions  $Oe_2$ ,  $Oe_3$ ,  $Oe_4$  (Fig. 19); then  $Oe_2$ ,  $Oe_3$ ,  $Oe_4$  are by the symmetry also parallel to possible edges. Hence  $Oe_1e_3$ ,  $Oe_2e_4$  are parallel to possible faces, and therefore their intersection  $OB$  is parallel to a possible edge.

Again  $Oe_1e_4$ ,  $Oe_2e_3$  are parallel to possible planes and meet in a line perpendicular to  $OB$  which is parallel to a possible edge; and the same holds for  $Oe_1e_2$ ,  $Oe_3e_4$ .

Hence the plane perpendicular to  $OB$  contains two lines parallel to possible edges and is therefore parallel to a possible face.

( $\gamma$ ). An exactly similar proof holds for  $n = 6, 8, 10$ , &c.



Fig. 19.

Case (2).  $n$ -al axes of the first sort;  $n$  odd.

(a).  $n = 5$ . Let  $Oe_1$  be parallel to any possible edge,

and let rotations about  $OB$  through  $\frac{2\pi}{5}$  bring  $Oe_1$  successively to  $Oe_2$ ,  $Oe_3$ ,  $Oe_4$ ,  $Oe_5$ : these lines are also parallel to possible edges (Fig. 20).

\*  $O$  is below  $B$ , so that  $OB$  is perpendicular to the plane of the paper.

Then the planes  $Oe_1e_3$ ,  $Oe_3e_4$  are parallel to possible faces, and therefore meet in a line  $Ol_2$  parallel to a possible edge; similarly  $Oe_2e_3$ ,  $Oe_4e_5$  meet in a line  $Ol_1$ , parallel to a possible edge; hence  $Oe_1l_1$ ,  $Oe_3l_2$  are parallel to possible faces, and their intersection  $OB$  is parallel to a possible edge.

Again the planes  $Oe_1e_3$ ,  $Oe_4e_5$ ;  $Oe_2e_3$ ,  $Oe_3e_4$  are parallel to possible faces; the first two meet in a line perpendicular to  $OB$ , and so do the last two.

Hence the plane perpendicular to  $OB$  is parallel to two possible edges and is therefore parallel to a possible face.

( $\beta$ ). An exactly similar proof holds for  $n = 7, 9, 11$ , &c.

( $\gamma$ ).  $n = 3$ . In this case the theorem cannot be proved by aid of the law of rational indices alone; a

proof will be given in Part II\* with the help of the structure theory. The reason of the failure will be seen from the following discussion.

Let  $OX$  be parallel to any possible edge, and let rotations through  $\frac{2\pi}{3}$  and  $\frac{4\pi}{3}$  about the axis of symmetry bring  $OX$  to  $OY$  and  $OZ$ ; these lines are also parallel to possible edges. Take  $OX, OY, OZ$  as crystallographic axes. Let the Cartesian coordinates of any point referred to these lines as axes of reference be  $x, y, z$ ; and let the equation of any plane parallel to a crystal face be  $Hx + Ky + Lz = 0$ . Then by the symmetry  $Kx + Ly + Hz = 0$  and  $Lx + Hy + Kz = 0$  must also be parallel to crystal faces. The intercepts of these three faces on the crystallographic axes are inversely proportional to

$$1, \frac{K}{H}, \frac{L}{H}; \quad 1, \frac{L}{K}, \frac{H}{K}; \quad 1, \frac{H}{L}, \frac{K}{L}, \text{ respectively.}$$

\* Pp. 144, 256.

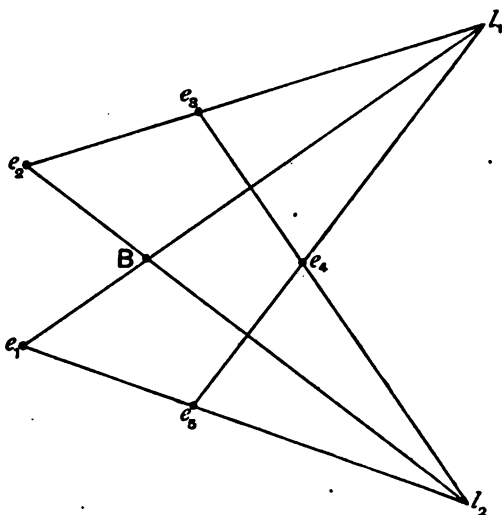


Fig. 20.

Therefore by the law of rational indices, since  $1:1:1$  are rational ratios,

$$\frac{K}{H} : \frac{L}{K} : \frac{H}{L} \text{ and } \frac{L}{H} : \frac{H}{K} : \frac{K}{L}$$

must be also rational, i. e.

$$\frac{K^2}{LH} : 1 : \frac{HK}{L^2} \text{ and } \frac{LK}{H^2} : 1 : \frac{K^2}{LH}$$

must be rational ratios.

Hence 
$$\frac{H^2}{KL}, \frac{K^2}{LH}, \frac{L^2}{HK}$$

are rational, and therefore  $H^3:K^3:L^3$  are rational ratios.

But to show that the plane  $x+y+z=0$  (perpendicular to the axis of symmetry) is parallel to a possible face, we must know that  $H:K:L$  are rational ratios\*, which we cannot prove.

Note that if we know that the plane perpendicular to the symmetry-axis is parallel to a possible face, we may take its symbol as  $(111)$ ; and then the indices of the symmetry-axis are  $1, 1, 1$  which are rational, and hence the symmetry-axis is parallel to a possible edge.

Conversely, if the symmetry-axis  $OB$  is parallel to a possible edge we may take its symbol as  $[111]$ .

Then the axial ratios are  $1:1:1$ , for the direction-ratios of  $OB$  referred to  $OX, OY, OZ$  are evidently in the ratios  $1:1:1$  (p. 12).

Therefore the plane perpendicular to  $OB$  has the rational indices  $1, 1, 1$  and is parallel to a possible face.

There are some special cases of interest; for instance, we can prove the theorem if we know that there is a plane parallel to a crystal face, whose equation  $Hx+Ky+Lz=0$  is such that  $H^p:K^p$  is rational ( $p$  being an integer not divisible by 3).

For if  $p = 3m+1$  ( $m$  integral),  $\frac{H^{3m+1}}{K^{3m+1}}$  is rational and also  $\frac{H^{3m}}{K^{3m}}$  is rational (for  $\frac{H^3}{K^3}$  is rational).  $\therefore \frac{H^{3m+1}}{K^{3m+1}} + \frac{H^{3m}}{K^{3m}} = \frac{H}{K}$  is rational.

\* The intercepts made by a plane parallel to  $x+y+z=0$  on the axes are in the ratio  $1:1:1$ ; those made by a plane parallel to  $Hx+Ky+Lz=0$  are in the ratios,  $\frac{1}{H} : \frac{1}{K} : \frac{1}{L}$ ;  $\therefore$  by the law of rational indices, if both planes are parallel to possible faces,  $H:K:L$  are rational ratios.

Similarly if  $p = 3m - 1$ ,  $\frac{H^{3m}}{K^{3m}} + \frac{H^{3m-1}}{K^{3m-1}} = \frac{H}{K}$  is rational.

Now because  $Hx + Ky + Lz = 0$  is parallel to a possible face,  $\therefore$  by the law of rational indices  $Hx + Hy + Lz = 0$  is also parallel to a possible face, since  $H : K$  is rational. And by the symmetry  $Hx + Ly + Hz = 0$ ,  $Lx + Hy + Hz = 0$  are also parallel to possible faces.

Now these three planes meet in pairs in the lines  $Og_3$ ,  $Og_2$ ,  $Og_1$ , such that  $Og_1X$ ,  $Og_2Y$ ,  $Og_3Z$  are parallel to possible faces and are concurrent in the symmetry-axis, which is therefore parallel to a possible edge. By what we have proved above the perpendicular plane is parallel to a possible face.

Again we may prove the theorem if it is known that there are two lines through  $O$  which are parallel to possible edges and *equally inclined to the symmetry-axis*\*

For let  $Oe_1$ ,  $Of_1$  be these lines and let them be brought by rotations through  $\frac{2\pi}{3}$

about the symmetry-axis successively to the positions  $Oe_2$ ,  $Of_2$ ;  $Oe_3$ ,  $Of_3$  (Fig. 21). Then the plane passing through  $O$  and the intersections of the planes  $Oe_1f_2$ ,  $Of_1e_3$  and  $Oe_2f_3$ ,  $Of_2e_1$ , and the plane through  $O$  and the intersections of the planes  $Of_2e_1$ ,  $Oe_2f_3$  and  $Of_1e_3$ ,  $Oe_1f_2$  are parallel to possible faces and meet in the symmetry-axis. This axis is therefore parallel to a possible edge.

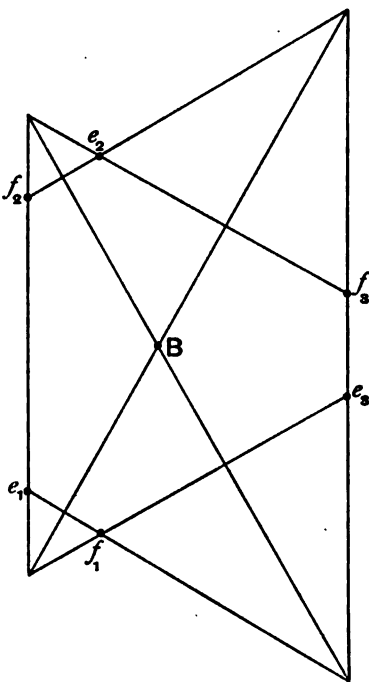


Fig. 21.

Case (3).  $n$ -al axes of the second sort;  $n$  even.

(a). In the case of  $n = 2$  the theorem has no meaning,

\* One of the lines can be brought into coincidence with the other by a rotation about the symmetry-axis through a certain angle. We assume that this angle is  $\neq 0$ ,  $\frac{2\pi}{3}$ , or  $\frac{4\pi}{3}$ .

for a 2-al axis of the second sort is merely a centre of symmetry.

( $\beta$ ).  $n > 2$ . Since an  $n$ -al axis of the second sort is an  $\frac{n}{2}$ -al axis of the first sort if  $n$  is even (§ 15, p. 36), every case except that of  $n = 6$  falls under case (1) ( $\alpha$ ), ( $\beta$ ), or ( $\gamma$ ), or under case (2) ( $\alpha$ ) or ( $\beta$ ).

Now a 6-al axis of the second sort is equivalent to a 3-al rotation-axis combined with a centre of symmetry (p. 36); this centre of symmetry can make no difference to the results discussed under case (2) ( $\gamma$ ); hence the theorem can no more be proved for a 6-al axis of the second sort than for a 3-al axis of the first sort.

Case (4).  $n$ -al axes of the second sort,  $n$  odd.

In this case the  $n$ -al axis of the second sort is equivalent to an  $n$ -al axis of the first sort combined with a *perpendicular symmetry-plane* (§ 15, p. 35); hence the theorem follows at once from (§ 17, p. 37).

It may be noticed that though we can find no proof of the above theorem in the case of a 3-al axis of the first sort or a 6-al axis of the second sort; yet in crystals of all substances hitherto investigated which have such an axis, either edges have been found which the most accurate measurements we can make cannot prove to be inclined to the axis of symmetry, or faces which cannot be shown not to be perpendicular to that axis.

*Corollary 1.* There are an indefinite number of possible edges perpendicular to any symmetry-axis  $b$ . For any possible face meets a plane perpendicular to  $b$  in a line parallel to a possible edge, since this plane is itself parallel to a possible face.

*Corollary 2.* If  $c$  be such a line through the centre  $O$  of the crystal perpendicular to an  $n$ -al axis  $b$  and parallel to a possible edge, and if  $c'$  be the position into which  $c$  would be brought by a rotation through  $\frac{2\pi}{n}$  about  $b$ ; then there is a possible face making equal intercepts on  $c$  and  $c'$ . ( $n > 2$ ).

For since  $b$  and  $c$  are parallel to possible edges, the plane through  $b$  and  $c$  is parallel to a possible face.

Let any other plane through  $O$  parallel to a possible face meet this plane in the line  $d$  which must be parallel to a possible edge.

First let  $b$  be a rotation-axis; and let  $d'$  be the position

into which  $d$  would be brought by a rotation through  $\frac{2\pi}{n}$  about  $b$ . Then since  $d$  is parallel to a possible edge, by the symmetry of the crystal  $d'$  must also be parallel to a possible edge.

Therefore there must be a possible face parallel to the lines  $d$  and  $d'$ , and this face makes equal intercepts on  $c$  and  $c'$ .

Next let  $b$  be an axis of the second sort; and let  $d'$  be the position into which  $d$  would be brought by a rotatory-reflexion through  $\frac{2\pi}{n}$  about  $b$ . Then, as before, there is a possible face parallel to the lines  $d$  and  $d'$ , and this face makes equal intercepts on  $c$  produced and on  $c'$ . Let the indices of this face referred to  $c$ ,  $c'$ , and  $b$  as crystallographic axes be  $hkl$ ; then there is a possible face  $(\bar{h}kl)$  making equal intercepts on  $c$  and  $c'$ .

§ 19. *The only possible symmetry-axes of a crystal are 2-al, 3-al, 4-al, and 6-al axes.*

Let  $b$  be an  $n$ -al symmetry-axis of a crystal (of the first or second sort), let  $c$  be a line through  $O$  the centre of the crystal perpendicular to  $b$  and parallel to a possible edge, and let  $c'$ ,  $c''$  (Fig. 22\*) be the positions into which  $c$  would be brought by rotations through  $\frac{2\pi}{n}$ ,  $\frac{4\pi}{n}$  about  $b$ . Then by the symmetry of the crystal  $c'$  and  $c''$  are also parallel to possible edges. Now there is a possible face making equal intercepts on  $c$  and  $c'$  (§ 18, Cor. 2); take this as parametral face, and take  $c$ ,  $c'$ ,  $b$  as crystallographic axes (this is lawful since these lines are parallel to possible edges).

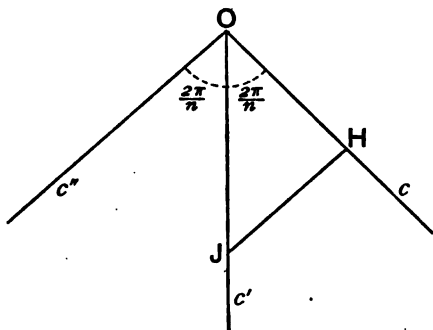


Fig. 22.

Since  $b$  and  $c''$  are parallel to possible edges, there is a possible face parallel to these two lines; let it meet  $c$  and  $c'$  in  $H$  and  $J$ . Then the indices of this face are proportional to  $\frac{1}{OH}$ ,  $\frac{1}{OJ}$ , 0; that is to  $2 \cos \frac{2\pi}{n}$ , 1, 0.

Since  $b$  and  $c''$  are parallel to possible edges, there is a possible face parallel to these two lines; let it meet  $c$  and  $c'$  in  $H$  and  $J$ . Then the indices of this face are proportional to  $\frac{1}{OH}$ ,  $\frac{1}{OJ}$ , 0; that is to  $2 \cos \frac{2\pi}{n}$ , 1, 0.

\*  $b$  is perpendicular to the plane of the paper in this figure.

Hence  $2 \cos \frac{2\pi}{n}$  is rational.

Now

$$\begin{aligned} \left(2 \cos \frac{2\pi}{n}\right)^n - a_1 \left(2 \cos \frac{2\pi}{n}\right)^{n-2} + a_2 \left(2 \cos \frac{2\pi}{n}\right)^{n-4} \\ - a_3 \left(2 \cos \frac{2\pi}{n}\right)^{n-6} + \dots = 2 \cos n \left(\frac{2\pi}{n}\right) = 2, \end{aligned}$$

where  $a_r = {}^{n-r}C_r + {}^{n-r-1}C_{r-1}$  and is integral\*.

Therefore  $2 \cos \frac{2\pi}{n}$  cannot be of the form  $\frac{p}{q}$  where  $p$  and  $q$  are integers prime to one another; for in that case we should

have 
$$\frac{p^n}{q^n} - a_1 \frac{p^{n-2}}{q^{n-2}} + a_2 \frac{p^{n-4}}{q^{n-4}} - \dots = 2$$

or 
$$\frac{p^n}{q} = 2q^{n-1} + a_1 p^{n-2} q - a_2 p^{n-4} q^3 + \dots,$$

a fraction equal to an integer, which is impossible.

But  $2 \cos \frac{2\pi}{n}$  is rational, and we have just shown that it is not fractional; therefore  $2 \cos \frac{2\pi}{n}$  is integral. Also  $2 \cos \frac{2\pi}{n}$  is numerically not greater than 2; therefore  $\cos \frac{2\pi}{n} = 0, \pm \frac{1}{2}$ , or  $\pm 1$ , whence we deduce  $n = 2, 3, 4, \text{ or } 6$ .

\* See Hobson's "Plane Trigonometry," p. 103.

## CHAPTER IV

## THE THEORY OF GROUPS.

§ 1. A series of operations is said to form a *group* when (1) the product of any two of the operations, or the square of any one, is equivalent to some member of the series\*, and (2) the series always contains the operation  $A^{-1}$  if it contains the operation  $A$ .

Every group contains the identical operation, for  $A \cdot A^{-1} = 1$ .

A group is said to be *finite* or *infinite* according as the number of operations it contains is finite or infinite.

*The product of any number of operations of a group is itself a member of the group.*

For if  $A.B.C.D \dots K$  is the product of any number of operations  $A, B, C, \dots$  of a group, not necessarily all distinct, then this product  $= M.C.D \dots K$  (where  $M$  is a member of the group);  $= N.D \dots K$  (where  $N$  is a member of the group);  $= H.K$  (where  $H$  is a member of the group, by repetition of the process);  $= J$  (where  $J$  is an operation of the group).

In particular any integral power of any operation of a group is a member of that group.

As an example of a group of operations we may take a rotation  $A$  through  $\pi$  about an axis  $a$ , a reflexion  $S$  in a plane  $\sigma$  perpendicular to  $a$ , and an inversion  $I$  about the intersection of  $a$  and  $\sigma$ . These movements form a group, for

$$A.S = S.A = I; I.S = S.I = A; I.A = A.I = S;$$

$$A^2 = S^2 = I^2 = 1.$$

§ 2. A group is said to be of the second or of the first sort according as it does or does not contain operations of the second sort.

Let  $L_1, L_2, L_3, L_4, \dots$  (i) be the distinct operations of the first sort of a group of the second sort, and let  $L', M$  be any two operations of the second sort contained in the group. Then  $L'^2$  and  $M^2$  are of the first sort†, and are therefore included in (i).

Now  $M.L'$  is an operation of the first sort† of the group; suppose  $M.L' = L_u$ .

Then  $M.L' . L_u^{-1} = 1$ ; hence  $L' . L_u^{-1} = M^{-1}$ , and therefore

\* i. e. some operation of the series.

† See p. 30.

$M = L' \cdot L_u^{-1} \cdot M^2 = L' \cdot L_p$ ; where  $L_p = L_u^{-1} \cdot M^2$  is included in (i).

Again  $L' \cdot L_p \neq L' \cdot L_j$ , for  $L_p \neq L_j$ ; hence we conclude that  $L' \cdot L_1, L' \cdot L_2, L' \cdot L_3, L' \cdot L_4, \dots$  (ii) are all distinct and include all operations of the second sort of the group,  $L'$  being any operation of the second sort contained in the group.

Similarly  $L_1 \cdot L', L_2 \cdot L', L_3 \cdot L', L_4 \cdot L', \dots$  (iii) are all distinct and include all operations of the second sort contained in the group. The series (iii) is the same as (ii), though the order of the operations in the two series is in general different.

It follows that:—

*Every finite group of the second sort contains as many operations of the first sort as of the second sort.*

§ 3. It sometimes happens that a certain portion of the operations of a group form a group when taken alone; the portion is then said to form a *subgroup*.

*If a group contains operations of the first and of the second sort; the operations of the first sort form a subgroup.*

For the product of two operations of the first sort is an operation of the first sort (p. 30).

§ 4. Let  $L_1, L_2, L_3, L_4, \dots$  (i) be the distinct operations of any subgroup  $G_1$ , of a group  $G$ .

Let  $M$  be any operation of  $G$  not contained in (i).

Then  $M \cdot L_1, M \cdot L_2, M \cdot L_3, M \cdot L_4, \dots$  (ii) are operations distinct from each other and from the operations (i). For  $M \cdot L_u$  cannot equal\*  $M \cdot L_v$  for then  $L_u$  would equal  $L_v$ , and  $M \cdot L_u$  cannot equal  $L_p$  for then  $M$  would equal  $L_p \cdot L_u^{-1}$ , that is, would equal some operation of the series (i).

Let  $M'$  be an operation of  $G$  not included in (i) and (ii) (if such an operation exists).

Then  $M' \cdot L_1, M' \cdot L_2, M' \cdot L_3, M' \cdot L_4, \dots$  (iii) are operations of  $G$  distinct from each other and from the operations (i) and (ii). For as before  $M' \cdot L_u$  cannot equal  $M' \cdot L_v$  or  $L_p$ , and it cannot equal  $M \cdot L_w$  for then  $M'$  would equal  $M \cdot L_w \cdot L_u^{-1}$ , that is, some operation of the series (ii).

If there is some operation of  $G$  not contained in (i), (ii), (iii), we proceed as before. If  $G$  is an infinite group it may or may not be possible to form an infinite number of rows similar to

\* Two operations are said to be 'equal' or 'equivalent' when they have identical effects on any figure.

(i), (ii), &c. If, however,  $G$  (and therefore  $G_1$ ) contains only a finite number of operations, we must at length reach a row ( $p$ ) such that it is impossible to find an operation of  $G$  different from all those in (i) (ii) ... ( $p$ ).

Now all the operations in these  $p$  rows are distinct and belong to  $G$ . Hence the number of operations of  $G$  is  $p$  times that of  $G_1$ , and therefore

*The number of operations of a finite group is a multiple of the number of operations of any subgroup.*

§ 5. Let  $a$  be any point, line, or plane, and let  $a$  be brought to coincidence with  $a_1, a_2, a_3, \dots$ , respectively, by the operations  $L_1, L_2, L_3, \dots$  of any group. Then  $a, a_1, a_2, a_3, \dots$  are said to be a system of *equivalent*\* points, lines, or planes. Any operation  $L_h$  of the group is a symmetry-operation of this system; for, if  $a_p$  is any member of the system and  $L_p \cdot L_h = L_e$ ,  $L_h$  brings  $a_p$  to coincide with  $a_e$  †.

Hence *the operations of any group are symmetry-operations of an infinite number of different figures.*

It should be noticed that  $a_p$  cannot be brought to coincide with  $a_e$  by two distinct operations  $L_h$  and  $L_k$ ; for then  $L_h \cdot L_k^{-1}$  would leave  $a_p$  unmoved, which is not in general the case unless  $L_h \cdot L_k^{-1} = 1$ , i.e.  $L_h = L_k$ . We deduce immediately (taking  $a_p$  as  $a$ ) that *the number of points, lines, or planes in an equivalent system of a finite group is in general equal to the number of operations in that group.*

§ 6. It follows from p. 33 that *the symmetry-operations of any figure form a group* ‡; for they satisfy the conditions of p. 45.

Let  $L$  be any symmetry-operation of a figure  $U$ , and let any symmetry-element  $b$  of  $U$  be brought into the position  $b'$  by  $L$ . Then, since  $U, b$  are brought into coincidence with  $U, b'$  respectively by  $L$ , therefore  $b'$  is related to  $U$  in the same way that  $b$  is. Therefore  $b'$  is a symmetry-element of  $U$  of the same kind as  $b$ . Hence *the symmetry-operations of any figure are symmetry-operations of the system formed by all the symmetry-elements of the same kind possessed by the figure.*

For example any symmetry-operation of  $U$  brings any  $n$ -al rotation-axis of  $U$  into coincidence with itself or some other

\* German—gleichwerthig.

†  $L_p^{-1}$  brings  $a_p$  to coincide with  $a$ , and  $L_e$  brings  $a$  to coincide with  $a_e$ ; therefore  $L_p^{-1} \cdot L_e$  brings  $a_p$  to coincide with  $a_e$ .

‡ This is the converse of the first theorem of the previous section.

$n$ -al rotation-axis of  $U$ , any symmetry-plane of  $U$  into coincidence with itself or some other symmetry-plane of  $U$ , &c.

This does not imply that any given  $n$ -al rotation-axis of  $U$  can be brought into coincidence with any other by a symmetry-operation of  $U$ , &c. A system of elements of the same kind such that any one can be brought into coincidence with any other by some symmetry-operation of  $U$  is called an 'equivalent system of symmetry-elements' of that kind.

By 'symmetry-elements of a group' we shall mean the axes, planes, &c., about which the operations of the group act; that is, the symmetry-elements of all figures which are brought to self-coincidence by every operation of the group. Similarly we talk of 'rotation-axes,' 'symmetry-planes,' &c. of a group. We say also that a group 'contains,' 'has,' or 'possesses' symmetry-elements.

§ 7. It is evident that a 'crystal' (see p. 36) can only be brought to self-coincidence in a finite number of ways; and therefore the symmetry-operations of a crystal form a *finite group*.

Such a group can only possess rotations about axes passing through a fixed point  $O$  (the 'centre' of the group), reflexions in planes through  $O$ , rotatory-reflexions whose axes meet the corresponding planes at  $O$ , and the inversion about  $O$ . All the operations of the group leave the point  $O$  unmoved.

Finite groups whose operations all leave one point unmoved are called *point-groups*. By 'group' we shall mean 'point-group' for the remainder of Part I.

§ 8. We proceed to prove certain properties of a system of equivalent points of a group (i.e. a point-group). Similar theorems will be seen to hold good for a system of equivalent lines or planes.

The number of points in the system is *in general* equal to the number of operations in the group (p. 47). If, however, one of the points has some special position, this is not necessarily true.

If one of the points of the system lies in a symmetry-element, every other point of the system lies in a symmetry-element of the same kind; for every operation of the group brings the system of symmetry-elements of the same kind to self-coincidence (p. 47).

Conversely if one of the points  $P$  does *not* lie in a symmetry-element of a given kind no other point  $P'$  of the system can do so; for if  $P'$  lay in a symmetry-element of the given kind so would  $P$ ; and this is contrary to hypothesis.

For example if one of the points of the system lies in an  $n$ -al rotation-axis so do all the rest; if one of the points is not in an  $n$ -al rotation-axis no other point can lie on such an axis.

§ 9. *If one of the points of an equivalent system lies in a symmetry-plane and in no other symmetry-element, the number of points in the system is half that of the operations of the group.*

For if one of the points lies in a symmetry-plane and in no other element, it coincides with its reflexion in that plane, but with no other point of the system. Moreover any other equivalent point also lies in a symmetry-plane and in no other element, and coincides with its reflexion in that plane, but with no other point of the system.

§ 10. If  $A$  is a rotation through  $\frac{2\pi}{n}$  about an  $n$ -al rotation-axis  $a$ , and  $P$  is a point on that axis, the  $n$  points to which  $P$  is brought by the operations  $1, A, A^2, \dots, A^{n-1}$  coincide. Also each of the points equivalent to  $P$  lies in an  $n$ -al rotation-axis equivalent to  $a$ ; hence it follows that

*If one of the points of an equivalent system lies in an  $n$ -al rotation axis, but in no other symmetry-element, the number of points in the system is  $\frac{1}{n}$  the number of operations in the group.*

Now suppose  $a$  lies in a symmetry-plane  $\sigma$ , and let  $S$  represent a reflexion in this plane; let  $P$  be a point on  $a$ . Then the  $2n$  points to which  $P$  is brought by the operations  $1, A, A^2, \dots, A^{n-1}, S, S.A, S.A^2, \dots, S.A^{n-1}$  coincide.

Also each of the equivalent points lies in an  $n$ -al rotation-axis equivalent to  $a$  and in a plane of symmetry equivalent to  $\sigma$ ; hence we have

*If one of the points of an equivalent system lies in an  $n$ -al axis of symmetry through which passes a plane of symmetry, the number of points in the system is  $\frac{1}{2n}$  the number of operations in the group.*

§ 11. A rotation-axis is said to be *two-sided* or *one-sided*\* according as points equidistant from and on opposite sides of the centre are or are not equivalent.

\* German—*zweiseitig, einseitig.*

A two-sided axis must evidently have a symmetry-plane or a  $2m$ -al axis perpendicular to it (where  $m$  is a positive integer).

*The number of operations in a group which contains a equivalent  $n$ -al rotation-axes is in general  $2na$  or  $na$ , according as the axes do or do not lie in symmetry-planes; where two-sided axes count for two separate axes in reckoning the number  $a$ .*

If a point  $P$  lies in one of the  $a$   $n$ -al axes, so do all points equivalent to  $P$ ; for the axes are equivalent. Therefore the number of points equivalent to  $P$  is in this case  $a$ . But (p. 49) this number is  $\frac{1}{2n}$  or  $\frac{1}{n}$  the number of operations in the group, according as the  $a$   $n$ -al axes do or do not lie in symmetry-planes. Therefore the number of operations is  $2na$  or  $na$  according as the  $a$  equivalent axes do or do not lie in symmetry-planes.

§ 12. *If the operations of the first sort of a group consist of the identical operation together with rotations about  $V_2$  2-al axes,  $V_3$  3-al axes,  $V_4$  4-al axes, ... (where a two-sided axis counts only as a single axis in reckoning  $V$ ); then the number of operations contained in the group is*

$$1 + V_2 + 2V_3 + 3V_4 + \dots$$

*or twice that number according as the group is of the first or second sort.*

For any point  $P$  is brought to coincide with  $(n-1)$  equivalent points by rotation about an  $n$ -al axis; hence if rotations are the only operations of the group there must be at least

$$V_2 + 2V_3 + 3V_4 + \dots$$






points equivalent to  $P$ , since it is impossible that  $P$  should be brought into coincidence with the same point by rotations through an angle less than  $2\pi$  about two different axes both passing through the centre of the crystal  $O$ . Moreover there cannot be more than  $V_2 + 2V_3 + 3V_4 + \dots$  points equivalent to  $P$ , for  $P$  is brought into coincidence with every equivalent point by a rotation about *some* axis belonging to the group. Hence if there are no operations of the second sort the whole number of operations including the identical operation is

$$1 + V_2 + 2V_3 + 3V_4 + \dots$$

If on the other hand the group contains operations of the second sort the whole number of operations is twice that of the operations of the first sort (p. 46). Now we have just

proved that the number of operations of the first sort is  $1 + V_2 + 2V_3 + 3V_4 + \dots$ , and therefore the whole number of operations is  $2(1 + V_2 + 2V_3 + 3V_4 + \dots)$ .

§ 13. The points of an equivalent system are evidently all equidistant from the centre of the group, and therefore all lie on a sphere. Consequently we may form a stereographic projection of the system, and such a projection often forms a useful representation of the group.

The projection of the great circle in which the sphere is cut by any symmetry-plane is denoted by a continuous line; and the projection of the great circle parallel to the plane of projection (if not a symmetry-plane) by a broken line. The projections of points in which the sphere is met by a 2-al, 3-al, 4-al, or 6-al rotation-axis are marked by the symbols , , , respectively; while the projections of points in which the sphere is met by a 4-al or 6-al axis of the second sort are marked by , , respectively.

## CHAPTER V

## FINITE GROUPS OF THE FIRST SORT.

§ 1. In this chapter we shall investigate those finite groups of the first sort—also called holoaxial groups—which contain only 2-al, 3-al, 4-al, and 6-al rotation-axes. No group of

symmetry-operations of a crystal whose faces obey the law of rational indices can contain rotation-axes other than these.

In each case we shall give a stereographic projection of a system of equivalent points (Figs. 23–31, and 83, 84).

The first group to be considered is that containing no operation of symmetry at all (Fig. 23).

The number of points in an equivalent system

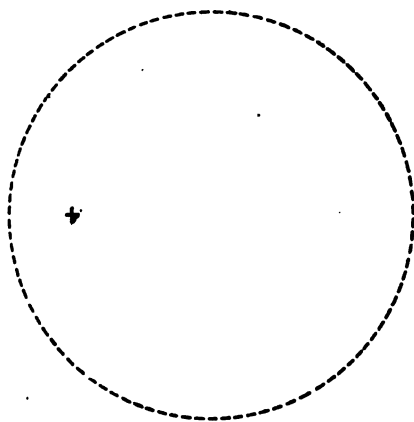


Fig. 23.  $C_1$ .

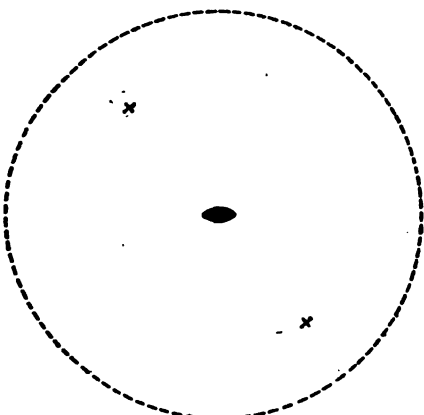
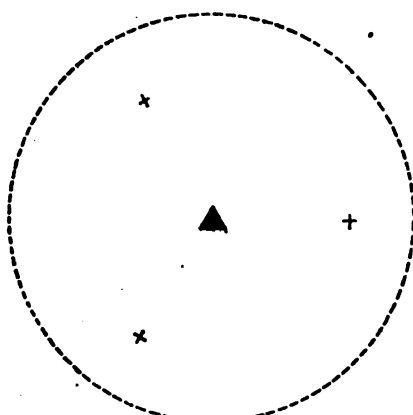
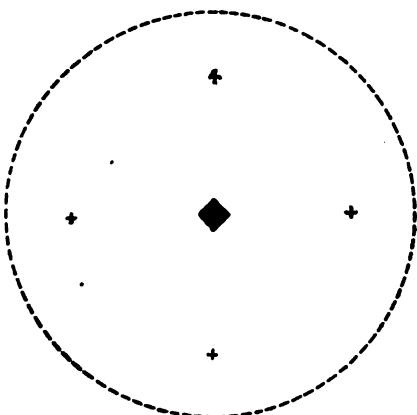
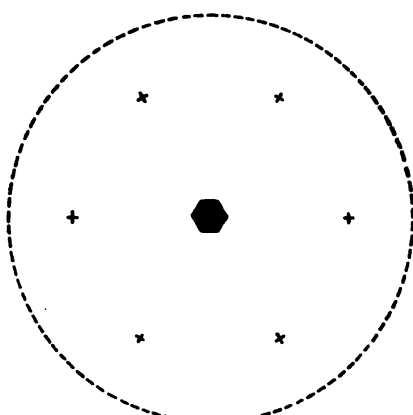
is one. We denote this group by the symbol  $C_1$ ; it contains only the identical operation.

§ 2. We have next the four groups with a single 2-al, 3-al, 4-al, or 6-al axis (Figs. 24–27; the axis is perpendicular to the plane of projection).

We denote these groups by the symbols  $C_2$ ,  $C_3$ ,  $C_4$ ,  $C_6$  respectively.

If  $A$  is a rotation through  $\frac{2\pi}{n}$  about the axis of symmetry the groups contain the operations  $1, A, \dots, A^{n-1}$  where  $n = 2, 3, 4$ , or  $6$ .

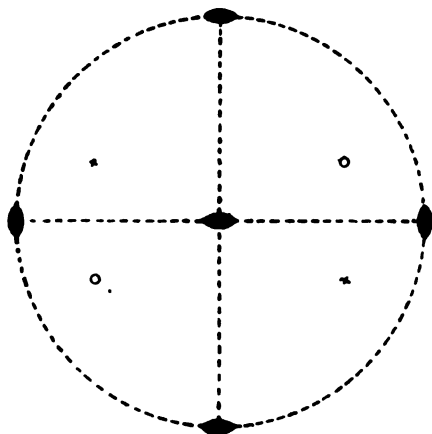
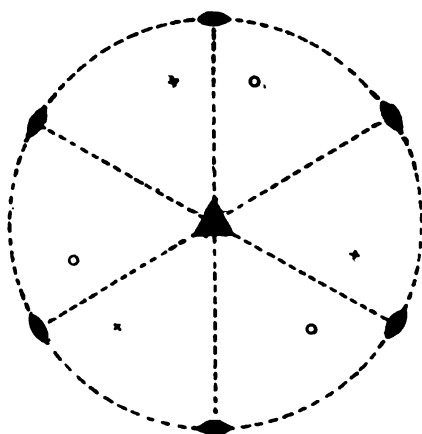
§ 3. Consider next a group containing one  $n$ -al axis  $a$  and a perpendicular 2-al axis  $b$ .

Fig. 24.  $C_2$ .Fig. 25.  $C_3$ .Fig. 26.  $C_4$ .Fig. 27.  $C_6$ .

Since each operation of the group brings the system of symmetry-elements into self-coincidence, the positions  $b', b'' \dots$  into which  $b$  would be brought by successive rotations through  $\frac{2\pi}{n}$  about  $a$ ,  $(n-1)$  in number if  $n$  is odd and  $\left(\frac{n}{2} - 1\right)$  in number if  $n$  is even, must be also occupied by 2-al axes of symmetry.

Now the resultant of the rotation through  $\frac{2\pi}{n}$  about  $a$  followed by the rotation through  $\pi$  about  $b$  is readily seen by

Euler's construction \* to be a rotation through  $\pi$  about a line  $c$  perpendicular to  $a$  and making an angle  $\frac{\pi}{n}$  with  $b$ . If  $n$  is odd  $c$  coincides with one of the series  $b, b', b'', \dots$ , but if  $n$  is even there must be a series of  $\frac{n}{2}$  2-al axes  $c, c', c'', \dots$  bisecting the angles between  $b, b', b'', \dots$ . In either case there can be no other symmetry-axis; for if  $a$  is the *only*  $n$ -al axis there can be no axes except  $a$  itself and 2-al axes perpendicular to  $a$ ; and there can be no such 2-al axes not belonging to the series  $b, b', b'', \dots$  †, or to the series  $c, c', c'', \dots$  ‡. For if there were such an axis let  $\beta$  be the angle between it and that axis of the series  $b, b', b'', \dots$  which makes the smallest angle with it; then since

Fig. 28.  $D_2 = Q$ .Fig. 29.  $D_3$ .

there are two 2-al axes perpendicular to  $a$  making an angle  $\beta$  with one another; therefore  $a$  is a  $\frac{\pi}{\beta}$ -al axis † contrary to hypothesis (for since  $\beta < \frac{\pi}{n}$ ,  $\therefore \frac{\pi}{\beta} > n$ ).

It is readily seen that the rotations about  $a$ ;  $b, b', b'', \dots$  †;  $c, c', c'', \dots$  ‡ form a group; and that  $b, b', b'', \dots$  are equivalent axes, and so are  $c, c', c'', \dots$ ; but that none of the  $b$ 's is equivalent to any one of the  $c$ 's (if  $n$  is even).

The total number of 2-al axes is always  $n$ , whether  $n$  is odd or even.

\* See p. 24.

† These two series coincide if  $n$  is odd.

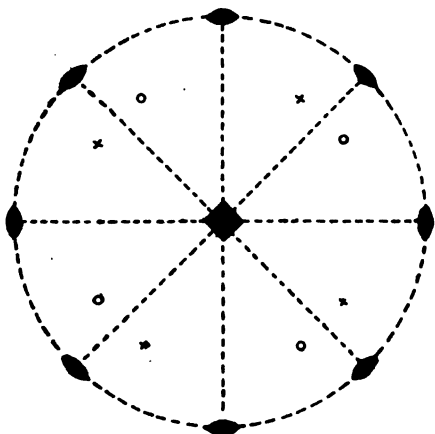
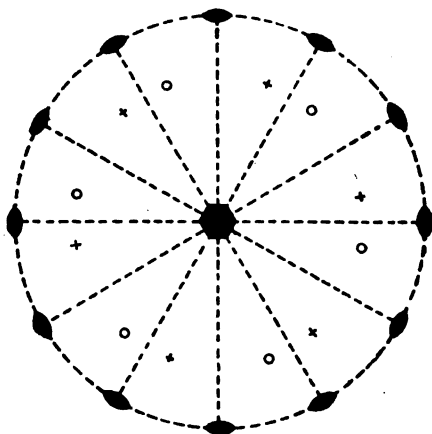
‡ This also follows from Euler's construction, see end of § 3, chap. iii, p. 25.

If  $A$  be the rotation through  $\frac{2\pi}{n}$  about  $a$ , and  $B$  the rotation through  $\pi$  about  $b$ , the operations of the group are evidently

$$\begin{array}{ccccccc} 1, & A, & A^2, & \dots, & A^{n-1}, \\ B, & A.B, & A^2.B, & \dots, & A^{n-1}.B. \end{array}$$

Taking  $n = 2$ , we have a group consisting of three mutually perpendicular 2-al axes (Fig. 28). This has been called the 'Quadratic' group\*. We shall represent it by  $Q$  or  $D_2$ †.

Taking  $n = 3, 4$ , and  $6$ , we have three groups which we shall denote by  $D_3, D_4$ , and  $D_6$ , respectively (Figs. 29–31; in these figures the axis  $a$  is perpendicular to the plane of pro-

Fig. 80.  $D_4$ .Fig. 81.  $D_6$ .

jection, and the lines in which planes through the vertex of projection and the 2-al axes cut the plane of projection are represented by broken lines).

§ 4. Consider now the most general case. Draw a sphere whose centre is  $O$ , the centre of the crystal. Let any  $n$ -al axis of the group meet the sphere in  $e$ , and let  $Od$  be any axis of the group which is not a 2-al axis perpendicular to  $Oe$ . Then the rotation about  $Od$  brings  $Oe$  into the position of some  $n$ -al

\* Introduced by G. G. Morrice as the English equivalent of the German 'Viererguppe' in his translation of F. Klein's "Lectures on the Icosahedron."

†  $D_2$  points out the analogy with  $D_3, D_4, D_6$ ;  $Q$  the fact that all its axes have similar properties.

axis equivalent to  $Oe$  (for the series of symmetry-elements is brought into self-coincidence by the rotation about  $Od$ ). Hence the group contains a series of equivalent  $n$ -al axes. Let  $Oe$  and  $Oe'$  be two of these axes, such that no two equivalent  $n$ -al axes of the group make an angle with each other smaller than the acute angle  $eOe'$ . Then if the rotation through  $\frac{2\pi}{n}$  about  $Oe'$  brings  $Oe$  into the position  $Oe''$  (Fig. 32),  $Oe''$  is an  $n$ -al

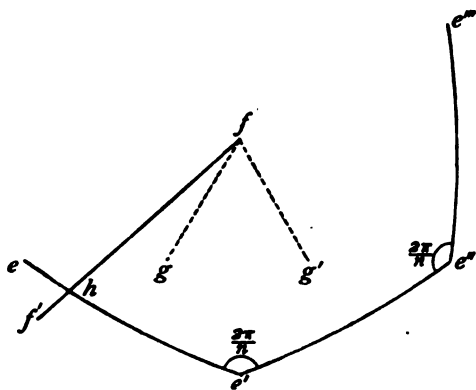


Fig. 32.

axis equivalent to  $Oe$  and  $Oe'$ . Similarly the rotation through  $\frac{2\pi}{n}$  about  $Oe''$  brings  $Oe'$  into the position of another equivalent  $n$ -al axis  $Oe'''$ , and proceeding in this way we see that the equivalent axes meet the sphere in points  $e, e', e'', \dots$  forming a regular spherical polygon. This polygon must

be closed, for otherwise there would be two equivalent axes of the system, making with each other an angle  $< eOe'$ . Let  $p$  be the number of sides of the polygon. The resultant of the rotation through  $\frac{2\pi}{n}$  about  $Oe$ , followed by an equal rotation about  $Oe'$ , is by Euler's construction a rotation about  $Of$  (where  $f$  is the pole of the small circle on which  $e, e', e'', \dots$  lie) through an angle  $-\frac{4\pi}{p}$ . Now the area

of the polygon  $ee'e'' \dots$  is  $[2n - p(n - 2)] \frac{K}{4n}$ , where  $K$  is the area of the whole surface of the sphere, and this must be positive.

The only possible values of  $n$  are 3, 4, 6 (rejecting  $n=2$  which gives us the group  $Q$  of § 3); the only values of  $p$  which make  $2n - p(n - 2)$  positive are

- for  $n=3$ ,  $p=2, 3, 4$  or  $5$ ;
- for  $n=4$ ,  $p=2$  or  $3$ ;
- for  $n=6$ ,  $p=2$ .

We reject  $p=2$  which gives us the groups of § 3, and also  $p=5$  which would make the line  $Of$  of Fig. 32 a 5-al axis and so inconsistent with the law of rational indices; we are left with

$$\begin{aligned} n=3, \quad p=3 \text{ or } 4; \\ n=4, \quad p=3. \end{aligned}$$

In the first case ( $n=3, p=3$ ) the polygon is an equilateral spherical triangle whose angles are  $\frac{2\pi}{3}$ , and whose sides are  $\cos^{-1}(-\frac{1}{2})$  (this is easily deduced by solving the spherical triangle  $efe'$  all of whose angles are known); we must reject this case, for since the angle  $eOe'$  is acute, its cosine cannot be negative.

In the second case ( $n=3, p=4$ ) the angle  $eOe'$  is seen by solving the triangle  $efe'$  to be  $\cos^{-1}\frac{1}{2}$ . This is the angle subtended at the centre of the sphere by a side of an inscribed cube. Hence if we produce  $eO, e'O, e''O, e'''O$  to meet the sphere again in  $e^{iv}, e^v, e^{vi}, e^{vii}$ , the eight points  $e, e', \dots, e^{vii}$  lie at the vertices of an inscribed cube. The group can contain no other 3-al axis; for evidently a rotation through  $\frac{2\pi}{3}$  about any line other than  $eO, e'O, e''O$ , or  $e'''O$ , does not bring this system of four equivalent 3-al axes to self-coincidence.

In the third case ( $n=4, p=3$ )  $eOe' = \frac{\pi}{2}$ , and hence  $eO, e'O, e''O$ , are mutually perpendicular, and if produced they cut the sphere at  $e''', e^{iv}, e^v$ . As before the group can contain no other 4-al axis.

§ 5\*. We can treat the problem in a more general way without assuming that  $n=2, 3, 4$ , or 6 as follows.

The rotation about  $Oe$  will bring the polygon  $ee'e'' \dots$  of Fig. 32 into a new position, and, by repeating the process of rotation about various equivalent axes, we can completely cover the surface of the sphere with polygons whose angular points lie on axes equivalent to  $Oe$ . Now these polygons do not overlap; for (1) if starting by rotating the polygon about  $Oe$  we could arrive at a polygon, one of whose angular points  $g$  lay inside  $ee'e'e'' \dots$ , but did not coincide with  $f$ ; by starting from a rotation about  $Oe'$  and pursuing a precisely similar course we could arrive at a polygon, one of whose angular points  $g'$  is situated relatively to  $e'$  as  $g$  is to  $e$  (Fig. 32). Then since  $fe=f'e', fg=fg'$ , and the angle  $efe'=gfg'$ ; but

\* §§ 5 and 6 may be omitted without affecting the subsequent argument.

$fg < fe < \frac{\pi}{2}$ ; therefore  $gg' < ee'$ ; contrary to the hypothesis that  $eOe'$  is the smallest angle between equivalent  $n$ -al axes of the group.

(2) If  $f$  is an angular point of one of the polygons and  $f'$  an angular point next to it; then  $f'$  cannot coincide with one of the points  $e, e', e'' \dots$ , for otherwise (since  $e'fe$  is then equilateral) the smallest angle of rotation about  $Oe$ , i.e. the angle  $2(fe'e')$ , would be  $> \frac{2\pi}{3}$ ; also  $f'$  cannot lie inside the polygon  $ee'e'' \dots$  as we have just proved, nor evidently on an edge of it. Suppose therefore that  $f'$  lies outside the polygon, and let  $ff'$  cut an edge (say  $ee'$ ) in  $h$ . Of  $eh$  and  $e'h$  not both can be the greater; suppose  $eh > e'h$  and similarly suppose  $f'h > fh$ . Then  $ef' < eh + f'h < \frac{1}{2}(ee' + ff') < ee'$  (since  $ee' = ff'$ ), i.e.  $Of'$  makes a smaller angle with  $Oe$  than  $Oe'$  does, contrary to hypothesis.

(3) It is impossible for one of the polygons to overlap  $ee'e'' \dots$  so that one of its angular points lies at  $e$  and an adjacent angular point lies at  $e''$  (say), for evidently  $ee'' > ee'$ .

The sphere then is completely covered with non-overlapping polygons whose vertices are the points where axes equivalent to  $Oe$  meet the sphere. Let  $S$  be the number of such points,  $E$  the number of polygon edges and  $F$  the number of the polygons.

Now the area of one of the polygons is  $[2n - p(n-2)] \frac{K}{4n} = \frac{K}{F}$ . Hence

$$F = \frac{4n}{2(p+n) - pn}.$$

As  $F$  is positive,  $2(p+n) > pn$  or  $\frac{1}{p} + \frac{1}{n} > \frac{1}{2}$ ; but (rejecting  $n$  or  $p = 2$  which leads to groups considered in § 3)  $\frac{1}{n}$  is  $> \frac{1}{3}$ ,

and therefore  $\frac{1}{p}$  must be  $< \frac{1}{6}$ ; similarly  $\frac{1}{n}$  must be  $< \frac{1}{6}$ . Hence  $n$  and  $p$  are both  $> 6$ .

It is found at once on trial that the only values of  $p$  and  $n$  which make  $F$  a positive integer are

$$\begin{aligned} n=3, & \quad p=3, 4 \text{ or } 5; \\ n=4 \text{ or } 5, & \quad p=3. \end{aligned}$$

We have also  $pF = nS = 2E$  for each quantity = the total

\* See § 4, p. 56.

number of polygon angles (such as  $e e' e''$ ), and therefore

$$S = \frac{4p}{2(p+n) - pn}, E = \frac{2pn}{2(p+n) - pn};$$

from these and  $F = \frac{4n}{2(p+n) - pn}$  we at once deduce the relation  $S + F = E + 2$ .

We have then the following table:—

$n$	$p$	$S$	$E$	$F$	<i>The axes meet the sphere at the vertices of a regular</i>
3	3	4	6	4	Tetrahedron
3	4	8	12	6	Hexahedron or Cube
4	3	6	12	8	Octahedron
3	5	20	30	12	Dodecahedron
5	3	12	30	20	Icosahedron

We see at once from the values of  $n$ ,  $p$ ,  $s$ ,  $E$  and  $F$  given in the table that the centres of the faces of a regular tetrahedron are the vertices of another regular tetrahedron; that the centres of the faces of a regular hexahedron are the vertices of a regular octahedron and vice versa; and that the same reciprocal relation is true of the dodecahedron and icosahedron.

§ 6. We proved in § 4 that the resultant of a rotation through  $\frac{2\pi}{n}$  about  $Oe$  followed by a similar rotation about  $Oe'$  is a rotation through  $\frac{-4\pi}{p}$  about  $Of$ . If  $p$  is odd this means that  $Of$  is a  $p$ -al axis; if  $p$  is even  $Of$  is either a  $\frac{p}{2}$ -al axis or a  $p$ -al axis. If therefore a group contains six 5-al axes passing through the vertices of a regular icosahedron it contains also ten 3-al axes passing through the vertices of a regular dodecahedron, and vice versa. The group also contains fifteen 2-al axes bisecting the edges of the icosahedron and dodecahedron, for the resultant of a rotation through  $\frac{2\pi}{5}$  about a 5-al axis followed by a rotation through  $\frac{2\pi}{3}$  in the same direction about a neighbouring 3-al axis is readily seen by Euler's construction to be a rotation through  $\pi$  about such a 2-al axis.

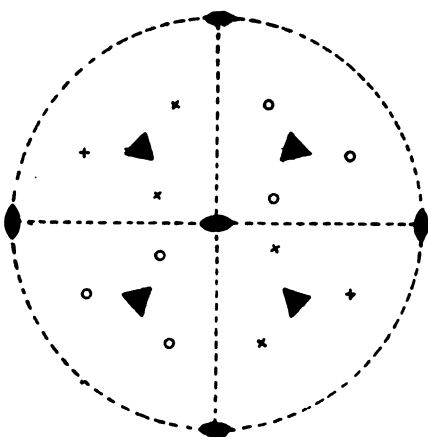
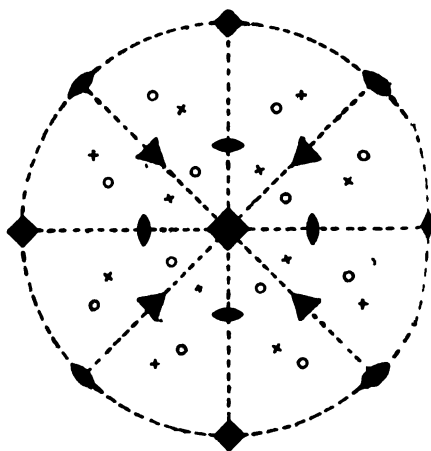
The rotations above-mentioned can easily be seen to form a group; it must contain sixty operations (p. 50). The group

contains 5-al axes, and therefore cannot be a group of symmetry-operations of a crystal obeying the law of rational indices.

§ 7. We now turn to the second case of § 4 ( $n = 3, p = 4$ ). In this case there are four 3-al axes lying along the diagonals of a cube, and the axes perpendicular to the faces of the cube are 2-al axes\* (for a rotation through  $\frac{4\pi}{p}$  about  $Of$  in Fig. 32 was proved in § 4 to be an operation of the group).

The rotations about these four 3-al axes and three 2-al axes are readily seen (by Euler's construction or otherwise) to form a group.

The system of equivalent points is shown in Fig. 33; we represent the group by the symbol  $T$ .

Fig. 33.  $T$ .Fig. 34.  $O$ .

§ 8. Consider now the third case of § 4 ( $n = 4, p = 3$ ). We have three 4-al axes lying along the diagonals of an octahedron, or, what is the same thing, perpendicular to the faces of a cube. These necessitate the existence of four 3-al axes perpendicular to the faces of the octahedron, or lying along the diagonals of the cube (for a rotation through  $(2\pi - \frac{4\pi}{p})$  about  $Of$  was proved in § 4 to be an operation of the group).

A rotation through  $\frac{2\pi}{p}$  about  $Of$  (Fig. 32) followed by a

\* Or may be 4-al; we shall return to this case later.

rotation in the same direction through  $\frac{2\pi}{n}$  about  $Oe$  is readily seen by Euler's construction to be equivalent to a rotation through  $\pi$  about an axis bisecting  $ee'$ ; hence in the case under consideration there must be six 2-al axes bisecting the angles between the 4-al axes (and also the angles between neighbouring 3-al axes).

The 4-al, 3-al, 2-al rotations above-mentioned are readily seen to form a group, which we shall represent by the symbol  $O$ .

If  $U, V, W$  denote rotations through  $\pi$  about the three 4-al axes,  $A$  a rotation through  $\frac{2\pi}{3}$  about a 3-al axis, and  $R$  a rotation through  $\pi$  about a 2-al axis, the operations of the group may be written

$$\begin{array}{llll} \{ \begin{array}{l} 1, \\ R, \\ A, \\ A.R, \\ A^2, \\ A^2.R, \end{array} & \begin{array}{l} U, \\ U.R, \\ U.A, \\ U.A.R, \\ U.A^2, \\ U.A^2.R, \end{array} & \begin{array}{l} V, \\ V.R, \\ V.A, \\ V.A.R, \\ V.A^2, \\ V.A^2.R, \end{array} & \begin{array}{l} W, \\ W.R, \\ W.A, \\ W.A.R, \\ W.A^2, \\ W.A^2.R. \end{array} \\ & & & \begin{array}{l} I \\ II \end{array} \end{array}$$

The first lines contain the operations of  $T$  which is a subgroup of  $O$ .

The projection of a system of equivalent points is given in Fig. 34.

§ 9. There are no other possible finite groups of the first sort. For in §§ 4 and 8 we proved that any group other than those of §§ 1, 2, 3, and 6 must contain four 3-al axes lying along the diagonals of a cube. Now any operation of a group containing these 3-al axes must bring these axes into self-coincidence and must therefore bring the cube of which they are the diagonals into self-coincidence; but evidently the only operations which can do this are the operations of the groups  $T$  and  $O$ .

NOTE.—The results of § 4 may be obtained without assuming the formula for the area of a spherical polygon. Since the angle  $ef'$  (Fig. 32) must have one of the values  $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}$ , we have only to see which of these values make  $\cos ee'$  positive and  $>1$ , when (1)  $fee' = \frac{\pi}{3}$ , (2)  $fee' = \frac{\pi}{4}$ ; given  $\sin^2 fee' \cdot \cos ee' = \cos^2 fee' + \cos ef'e'$ .

## CHAPTER VI

## FINITE GROUPS OF THE SECOND SORT.

§ 1. Since the operations of the first sort of a group  $G$  form a subgroup  $G_1$ ; and since the operations of the second sort of  $G$  can be found by multiplying the operations of  $G_1$  by *any* operation of the second sort ( $H$ ) belonging to  $G^*$ , and since the elements of symmetry are brought into self-coincidence by any operation of  $G$ ; therefore we can find the possible groups of the second sort by combining the operations of any group  $G_1$  of chapter v with an operation of the second sort which brings its system of rotation-axes into self-coincidence. It will be readily verified in each case that the operations so obtained really form a group †.

Since  $H^2$  is an operation of the first sort contained in  $G$ , therefore  $H^2$  is an operation of  $G_1$ .

It follows from p. 35 that if a group contains a 3-al axis of the second sort it contains a plane of symmetry; and that if it contains a 6-al axis of the second sort it contains a centre of symmetry. Hence the only operations of the second sort to be combined with the operations of the groups of chapter v, in order to get all the groups of the second sort, are a reflexion in some plane, an inversion about the centre

$O$ , and a rotatory-reflexion of angle  $\frac{\pi}{4}$  about some line through

$O$ . A rotation through  $\pi$  about this line is an operation of the group; the line must therefore coincide with a 2-al, 4-al, or 6-al rotation-axis. If, however, it coincides with a 4-al rotation-axis, the group must contain a reflexion; for

a rotatory-reflexion through  $\frac{\pi}{4}$  combined with a rotation through  $\frac{\pi}{4}$  in the opposite direction about the same line is equivalent to a reflexion in a plane perpendicular to that line.

\* P. 46.

† A proof (other than mere verification in each case) will be given for a more general case later; see p. 168.

If it coincides with a 6-al rotation-axis that axis is readily seen by similar reasoning to be a 12-al axis of the second sort, which is impossible in a group applicable to crystallography.

§ 2. We have first of all a group  $C_2$ , containing the two

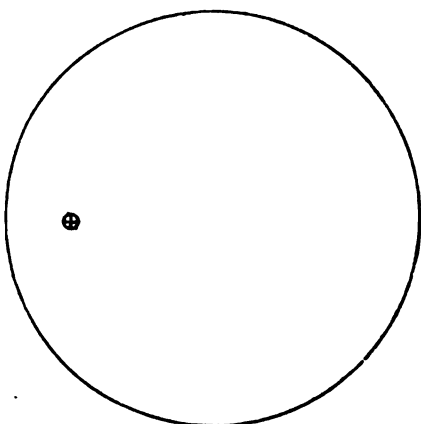


Fig. 85.  $C_2$ .

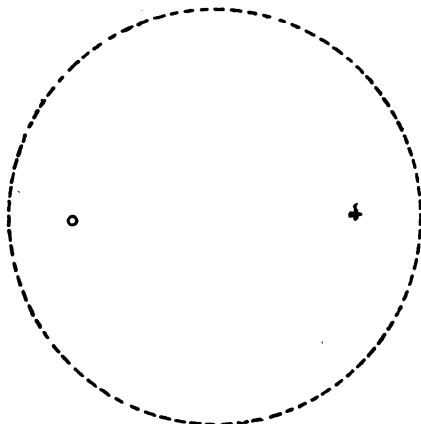


Fig. 86.  $C_4$ .

operations 1,  $S$ ; i. e. the identical operation and a reflexion in one plane of symmetry. (Fig. 85).

Next we have a group  $C_4$  containing the two operations 1,  $I$ ; i. e. the identical operation and inversion about a centre of symmetry. (Fig. 86). This group is that containing a 2-al axis of the second sort.

Then we have a group  $C_4'$  containing the operations 1,  $A'$ ,  $A'^2$ ,  $A'^3$ , where  $A'$  is a rotatory-reflexion about a single 4-al axis of the second sort. (Fig. 87).

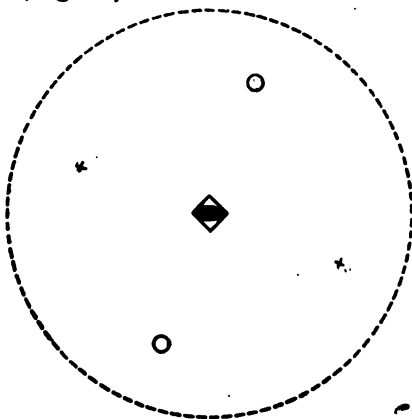


Fig. 87.  $C_4'$ .

§ 3. The rotation-axis of the groups  $C_2$ ,  $C_3$ ,  $C_4$ ,  $C_6$  is brought into self-coincidence by a reflexion  $S$  in a plane perpendicular

## 64 FINITE GROUPS OF THE SECOND SORT

to the axis. We thus have four groups  $C_{2h}$ ,  $C_{3h}$ ,  $C_{4h}$ ,  $C_{6h}$  (Figs. 38–41) whose operations are

$$\left. \begin{array}{l} 1, A, A^2, \dots, A^{n-1} \\ S, A.S, A^2.S, \dots, A^{n-1}.S \end{array} \right\},$$

or, writing them in a different order,

$$\left. \begin{array}{l} 1, A, A^2, \dots, A^{n-1} \\ S, S.A, S.A^2, \dots, S.A^{n-1} \end{array} \right\},$$

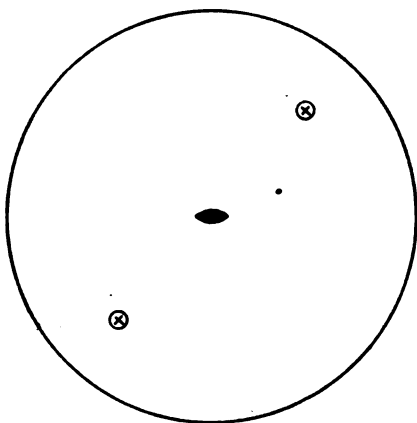


Fig. 38.  $C_{2h}$ .

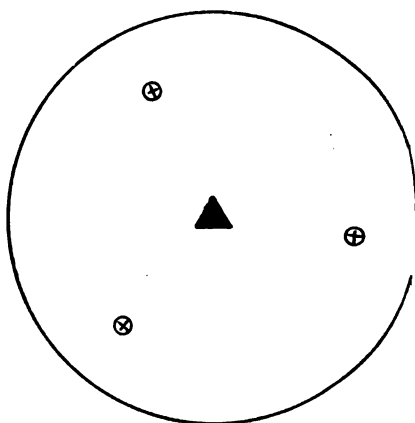


Fig. 39.  $C_{3h}$ .

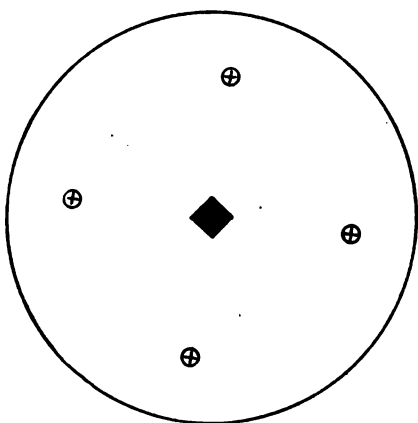


Fig. 40.  $C_{4h}$ .

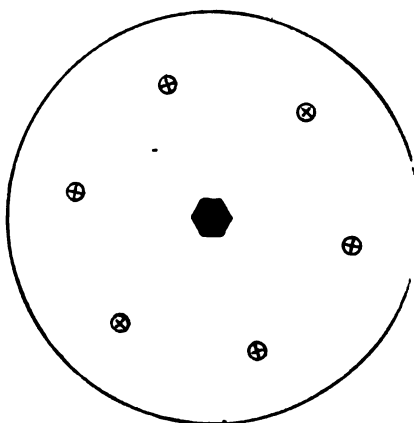


Fig. 41.  $C_{6h}$ .

where  $n = 2, 3, 4$ , or  $6$  (cf. chap. v, § 2, p. 52); and which contain one rotation-axis and a perpendicular plane of

symmetry. The groups  $C_{2h}$ ,  $C_{4h}$ ,  $C_{6h}$  contain a centre of symmetry.

§ 4. The rotation-axis of the groups  $C_2$ ,  $C_3$ ,  $C_4$ ,  $C_6$  is also brought into self-coincidence by a reflexion  $S_1$  in a plane  $s_1$  passing through the axis.

Let successive rotations through  $\frac{2\pi}{n}$  about the  $n$ -al axis

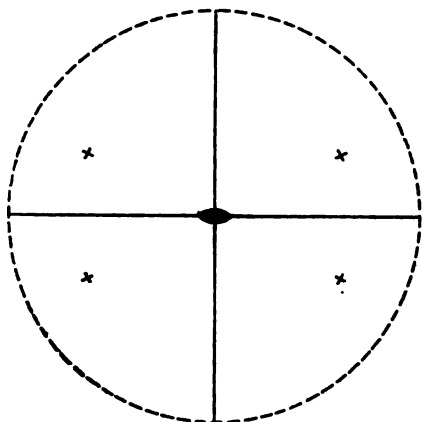


Fig. 42.  $C_{2v}$ .

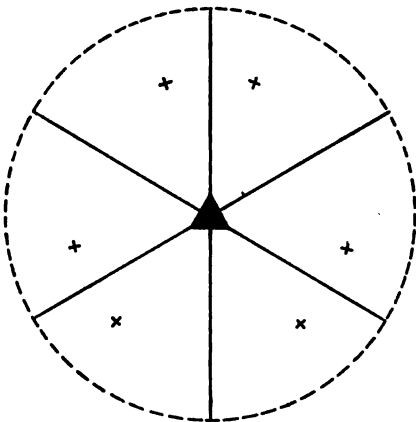


Fig. 43.  $C_{3v}$ .

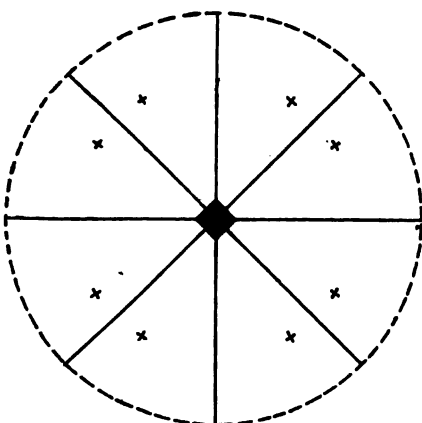


Fig. 44.  $C_{4v}$ .

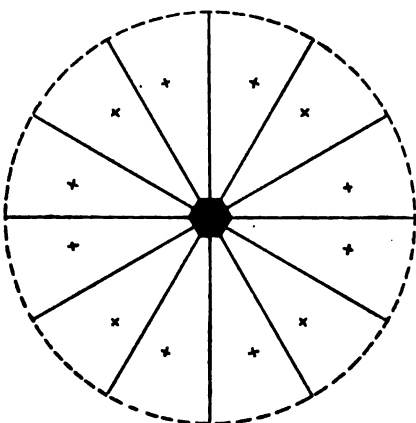


Fig. 45.  $C_{6v}$ .

of a group  $C_n$  bring  $s_1$  into the positions  $s_1', s_1'', \dots, (n-1)$  in number if  $n$  is odd and  $(\frac{n}{2} - 1)$  in number if  $n$  is even; then these planes are also planes of symmetry. The rotation

through an angle  $\frac{2\pi}{n}$  about the axis, followed by a reflexion in  $s_1$ , is equivalent to a reflexion in a plane  $\sigma_1$  making an angle  $\frac{\pi}{n}$  with  $s_1$ . (Cf. chap. iii. § 5, p. 26.)

If  $n$  is odd  $\sigma_1$  coincides with one of the series  $s_1, s'_1, s''_1, \dots$ , but if  $n$  is even there is a series of  $\frac{n}{2}$  symmetry-planes  $\sigma_1, \sigma'_1, \sigma''_1, \dots$  belonging to the group and bisecting the angles between the planes  $s_1, s'_1, s''_1, \dots$ .

There can be no other plane of symmetry passing through the axis. This may be proved by a method similar to that employed on p. 54.

It is readily seen that the planes  $s_1, s'_1, s''_1, \dots$  \* are equivalent, and so are the planes  $\sigma_1, \sigma'_1, \sigma''_1, \dots$  \*, but that none of the  $s_1^*$  are equivalent to any of the  $\sigma_1^*$  (if  $n$  is even).

The total number of symmetry-planes (whether  $n$  is odd or even) is  $n$ .

The operations of the group are

$$\left. \begin{array}{l} 1, A, A^2, \dots, A^{n-1} \\ S_1, A.S_1, A^2.S_1, \dots, A^{n-1}.S_1 \end{array} \right\}.$$

Taking  $n = 2, 3, 4$ , and  $6$  we have four groups  $C_{2v}, C_{3v}, C_{4v}, C_{6v}$  (Figs. 42, 43, 44, 45.)

No reflexion can bring a single rotation-axis to self-coincidence except those in planes passing through or perpendicular to that axis.

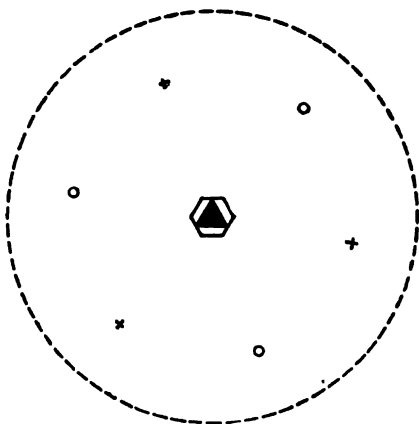


Fig. 46.  $C_{3i}$ .

§ 5. A centre of symmetry is an operation bringing the axis of a group  $C_n$  into self-coincidence, but it creates only the group  $C_{ni}$  if  $n$  is even.

If  $n = 3$  we have a new group  $C_{3i}$  (Fig. 46) whose operations are

$$\left. \begin{array}{l} 1, A, A^2 \\ I, A.I, A^2.I \end{array} \right\}.$$

This group is that containing a 6-al axis of the second sort (p. 36).

\* These two series coincide if  $n$  is odd.

A rotatory-reflexion of angle  $\frac{\pi}{2}$  only brings the axis of  $C_n$  into self-coincidence if its axis coincides with the axis of the rotatory-reflexion.

If  $n = 2$  this leads us again to  $C_4'$ ; we have already proved that if  $n = 4$  or  $6$  we can obtain no new group applicable to crystallography by combining the operations of  $C_n$  with a rotatory-reflexion of angle  $\frac{\pi}{2}$ .

§ 6. A reflexion  $S$  in a plane perpendicular to the  $n$ -al axis of the group  $D_n$  brings the system of rotation-axes to self-

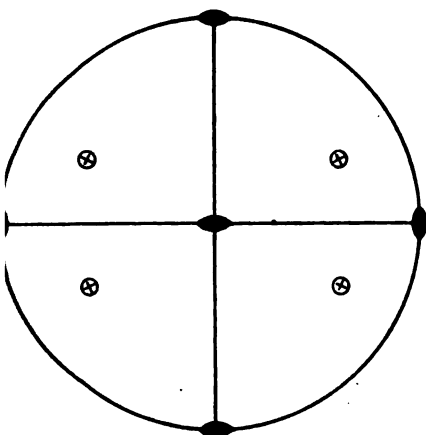


Fig. 47.  $D_{2h}$  or  $Q_4$ .

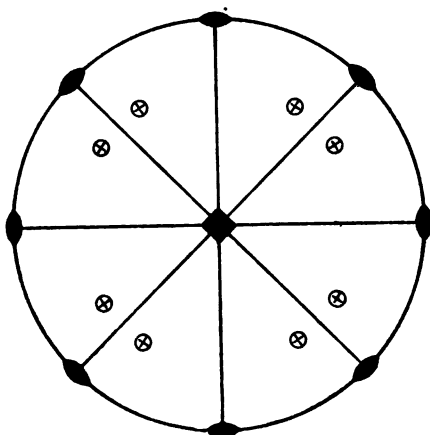


Fig. 48.  $D_{4h}$ .

coincidence; we get thus a group of the second sort  $D_{nh}$ ; whose operations are

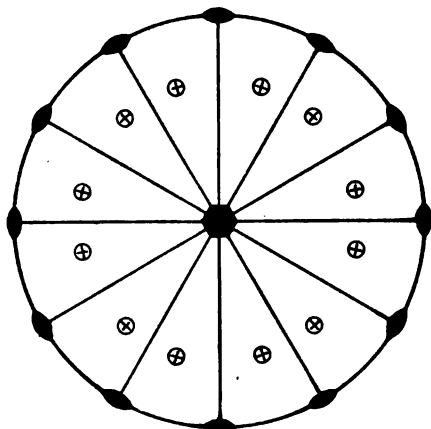
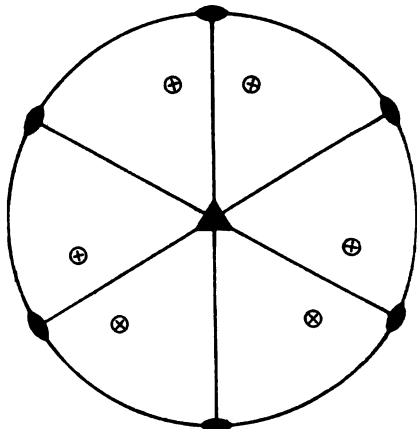
$$\begin{array}{llll} 1, & A, & A^2, & \dots, A^{n-1}; \\ S, & A.S, & A^2.S, & \dots, A^{n-1}.S; \\ B, & A.B, & A^2.B, & \dots, A^{n-1}.B. \\ B.S, & A.B.S, & A^2.B.S, & \dots, A^{n-1}.B.S. \end{array}$$

If  $n$  is even the group  $D_{nh}$  must possess a centre of symmetry; in this case it must have also  $n$  planes of symmetry perpendicular to the  $n$  2-al axes; each of these planes passes through the  $n$ -al axis and one of the 2-al axes.

Systems of equivalent points for the groups  $D_{2h}$ \*,  $D_{4h}$ ,  $D_{6h}$  are shown in Figs. 47, 48, and 49.

\* Also called  $Q_h$  when we wish to call attention to the fact that all its axes have similar properties.

If  $n$  is odd we have no centre of symmetry, but since a 2-al axis and a symmetry-plane passing through it involve another symmetry-plane perpendicular to the first (chap. iii. § 5, p. 26), the group  $D_{3k}$  (Fig. 50) contains three symmetry-

Fig. 49.  $D_{6k}$ .Fig. 50.  $D_{3k}$ .

planes each passing through the 3-al axis and one of the 2-al axes.

It may be seen that the  $n$ -al axis of the first sort of the group  $C_{nh}$  or  $D_{nh}$  is also an  $n$ -al axis of the second sort.

§ 7. The resultant of a reflexion in a plane  $s$ , followed by a rotation through  $\pi$  about an axis  $b$  making an angle  $a$  with  $s$ , is the resultant of a reflexion in  $s$  followed by a reflexion in the plane perpendicular to  $b$  (which makes an angle  $\frac{\pi}{2} - a$  with  $s$ ) and an inversion about the point  $O$  where  $b$  and  $s$  meet.

This is equivalent to a rotation through  $(\pi - 2a)$  about an axis  $c$  lying in  $s$  and perpendicular to  $b$  followed by an inversion about  $O$ , i. e. to a rotation through  $(\pi - 2a)$  about  $c$  followed by a rotation through  $\pi$  about  $c$  and a reflexion in the plane through  $O$  perpendicular to  $c$ , or to a rotatory-reflexion through an angle  $-2a$  about  $c$ .

Hence if a group contains a plane of symmetry and a 2-al axis making an angle  $\frac{\pi}{m}$  with it, it contains an  $m$ -al axis of the second sort lying in the plane of symmetry and perpendicular to the 2-al axis.

§ 8. A reflexion  $S_d$  in a symmetry-plane  $s_d$  bisecting the angles between the 2-al axes of the group  $D_n$  brings the system of axes of that group into self-coincidence; and hence creates a group of the second sort  $D_{nd}$  when combined with the rotations about these axes.

The operations of this group are

$$\begin{array}{lllll} 1, & A, & A^2, & \dots, & A^{n-1}; \\ S_d, & A.S_d, & A^2.S_d, & \dots, & A^{n-1}.S_d; \\ B, & A.B, & A^2.B, & \dots, & A^{n-1}.B. \\ B.S_d, & A.B.S_d, & A^2.B.S_d, & \dots, & A^{n-1}.B.S_d. \end{array}$$

By § 7 this group contains a  $2n$ -al axis of the second sort and therefore it is inconsistent with the law of rational indices unless  $n = 2$  or  $3$ .

Systems of equivalent points for these two cases are given in Figs. 51 and 52.

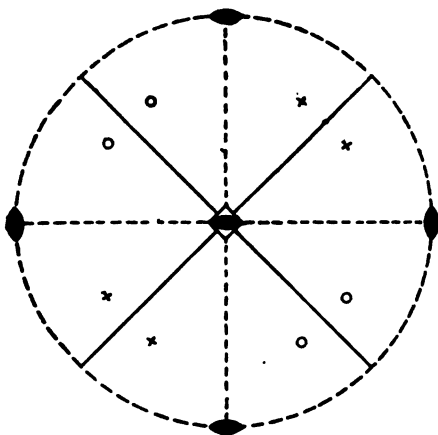


Fig. 51.  $D_{2d}$ .

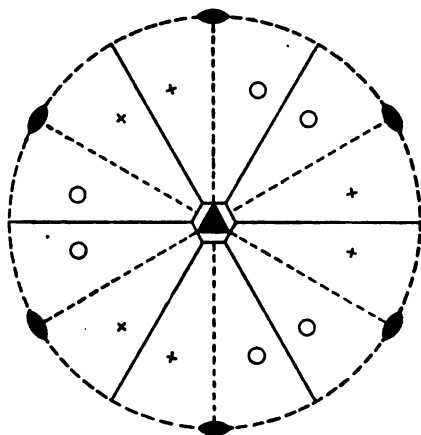


Fig. 52.  $D_{3d}$ .

In the case of  $D_{3d}$  the symmetry-planes are perpendicular to the 2-al axes, and there is therefore a centre of symmetry.

§ 9. A centre of inversion brings the axes of  $D_n$  into self-coincidence, but we get no *new* group by combining a centre of inversion with these axes.

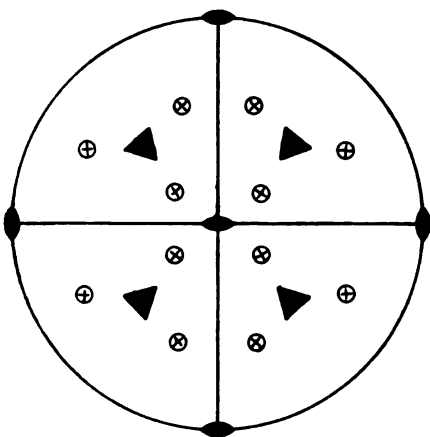
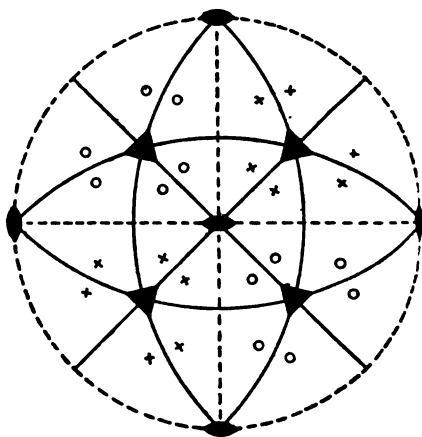
A rotatory-reflexion of angle  $\frac{\pi}{2}$  only brings the system of axes of  $D_n$  into self-coincidence, if its  $n$ -al axis coincides with the axis of rotatory-reflexion.

If  $n = 2$  this leads us again to  $D_{2d}$ ; we have already proved

that if  $n = 4$  or  $6$  we can obtain no new group applicable to crystallography by combining the operations of  $D_n$  with a rotatory-reflexion of angle  $\frac{\pi}{2}$ .

§ 10. The system of axes of the group  $T$  is brought to self-coincidence by a reflexion  $S$  in a plane  $s$  passing through a pair of the 2-al axes.

The group formed by combining  $S$  with the operations of  $T$  must contain (p. 26) symmetry-planes through these two 2-al axes and perpendicular to the plane  $s$ ; i.e. the group contains symmetry-planes perpendicular to each of the three 2-al axes, and has therefore a centre of symmetry.

Fig. 53.  $T_h$ .Fig. 54.  $T_d$ .

A system of equivalent points of this group  $T_h$  is shown in Fig. 53.

Exactly similarly we may derive a group of the second sort from the operations of  $T$  and a reflexion  $S_1$  in a plane joining two adjacent 3-al axes.

Since a symmetry-plane passes through a 3-al rotation-axis the group must also possess two other symmetry-planes making angles of  $\frac{\pi}{3}$  with this plane and passing through the same 3-al axis; hence the six planes which pass through any pair of the four 3-al axes must all be symmetry-planes.

This group is denoted by  $T_d$  (Fig. 54); it has no centre of symmetry.

Since there are only two groups of the second sort which can be derived from  $D_2$ , i. e.  $D_{2h}$  and  $D_{2d}$ , and since  $D_2$  is a subgroup of  $T$ ; only two groups of the second sort, i. e.  $T_h$  and  $T_d$  can be derived from  $T$ .

§ 11. For the same reason two groups of the second sort *at most* can be derived from  $O$ ; and in fact the two groups obtained by combining the operations of  $O$  with  $S$  or  $S_1$  are identical. For the group with the operation  $S$  has a centre of symmetry, and therefore this group has a reflexion in a plane perpendicular to any 2-al axis of  $O$ ; i. e. it contains the operation  $S_1$ .

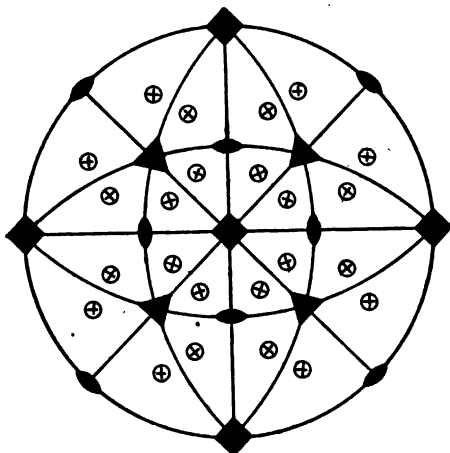


Fig. 55.  $O_h$ .

The operations of this group  $O_h$  (Fig. 55) may be written

$1,$	$U,$	$V,$	$W;$
$R,$	$U.R,$	$V.R,$	$W.R;$
$S,$	$U.S,$	$V.S,$	$W.S;$
$R.S,$	$U.R.S,$	$V.R.S,$	$W.R.S;$
$A,$	$U.A,$	$V.A,$	$W.A;$
$A.R,$	$U.A.R,$	$V.A.R,$	$W.A.R;$
$A.S,$	$U.A.S,$	$V.A.S,$	$W.A.S;$
$A.R.S,$	$U.A.R.S,$	$V.A.R.S,$	$W.A.R.S;$
$A^2,$	$U.A^2,$	$V.A^2,$	$W.A^2 \dots I$
$A^2.R,$	$U.A^2.R,$	$V.A^2.R,$	$W.A^2.R \dots II$
$A^2.S,$	$U.A^2.S,$	$V.A^2.S,$	$W.A^2.S \dots III$
$A^2.R.S,$	$U.A^2.R.S,$	$V.A^2.R.S,$	$W.A^2.R.S \dots IV$

The operations of  $T_i$  are contained in lines I and III, those of  $T_e$  in lines I and IV.

It is important to notice that the operations of  $C_{2h}$ ,  $D_{2h}$ ,  $C_{4h}$ ,  $D_{4h}$ ,  $D_{3d}$ ,  $C_{6h}$ ,  $D_{6h}$ ,  $T_h$ ,  $O_h$  may be obtained by combining the operations of  $C_2$ ,  $D_2$ ,  $C_4$ ,  $D_4$ ,  $D_3$ ,  $D_6$ ,  $T$ ,  $O$  respectively with an inversion. It would be better perhaps to denote  $D_{2h}(Q_h)$ ,  $T_h$ ,  $O_h$  by the symbols  $Q_i$ ,  $T_i$ ,  $O_i$  respectively, but to avoid confusion Schoenflies' symbols are retained.

§ 12\*. Only one group of the second sort can be derived from the group of § 6, chap. v; it is of course not applicable to crystallography. It has the fifteen planes joining opposite (and parallel) edges of the dodecahedron or icosahedron as symmetry-planes, and possesses a centre of symmetry.

\* This section may be omitted without affecting the subsequent argument.

## CHAPTER VII

## THE COORDINATES OF EQUIVALENT POINTS.

§ 1. We have now proved that there are thirty-two and only thirty-two finite groups of movements which are consistent with the law of rational indices and are therefore applicable to crystallography.

We may arrange them as follows:—

- I. 5 groups with more than one  $n$ -al axis ( $n > 2$ ).  
 $T, T_h, T_d, O, O_h \dots$  Regular system.
- II. 7 groups with a 6-al axis of the second or first sort.  
 $C_{3d}, D_{3d}; C_6, C_{6h}, C_{6v}, D_6, D_{6h} \dots$  Hexagonal system.
- III. 7 groups with a 4-al axis of the second or first sort.  
 $C'_4, D_{2d}; C_4, C_{4h}, C_{4v}, D_4, D_{4h} \dots$  Tetragonal system.
- IV. 5 groups with a 3-al axis of the first sort.  
 $C_3, C_{3h}, C_{3v}, D_3, D_{3h} \dots$  Trigonal system.
- V. 6 groups with a 2-al axis of the second or first sort.  
 $C_2; C_2, C_{2h}, C_{2v}, D_2, D_{2h} \dots$  Digonal system.
- VI. 2 groups with a 1-al axis of the second or first sort.  
 $C_\infty; C_1 \dots$  Monogonal system.

§ 2. For reasons which will appear later it is often more convenient to arrange these groups as follows\*:—

- I. Triclinic system † . . . . .  $C_1, C_i$ .
- II. Monoclinic system ‡ . . . . .  $C_2, C_2, C_{2h}$ .
- III. Orthorhombic system . . . . .  $C_{2v}, D_2, D_{2h}$ .
- IV. Tetragonal system . . . . .
  - (a) First (sphenoidal) subdivision . . .  $C_4, D_{2d}$ .
  - (b) Second subdivision . . .  $C_4, C_{4h}, C_{4v}, D_4, D_{4h}$ .

\* This is the more usual arrangement.

† Also called 'Anorthic system' or 'Asymmetric system.'

‡ Also called 'Monosymmetric system.'

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### V. Hexagonal system

#### (a) First (rhombohedral) subdivision

$C_3, C_{3i}, C_{3v}, D_3, D_{3d}$ .

#### (b) Second (sphenoidal) subdivision . . . $C_{3h}, D_{3h}$ .

#### (c) Third subdivision . . . $C_6, C_{6h}, C_{6v}, D_6, D_{6h}$ .

### VI. Regular system \* . . . . . $T, T_h, T_d, O, O_h$ .

The first subdivision of the hexagonal system is often treated as a separate system under the name 'Rhombohedral system'; and we shall so consider it in Part II.

We shall find in §§ 11 to 16 the coordinates of a system of equivalent points for each of these groups, choosing the axes of coordinates as conveniently as possible; the origin  $O$  in each case being at the centre of the crystal. We shall, however, first show how these coordinates may be obtained when the axes are chosen arbitrarily, and for that purpose shall first prove some general properties of coordinates and transformation of coordinates.

§ 3. Let  $OX, OY, OZ$  making angles  $\alpha, \beta, \gamma$  with one

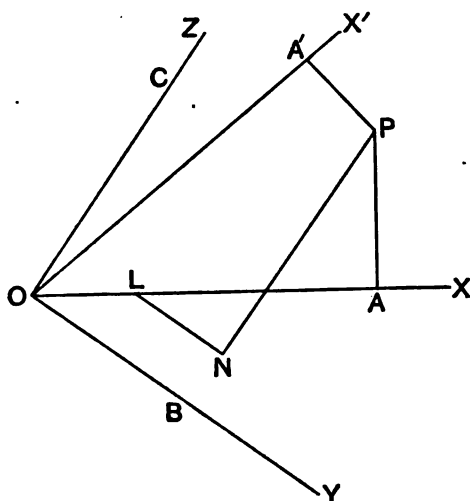


Fig. 56.

another be taken as axes of coordinates; we denote by  $x, y, z$  the Cartesian coordinates of any point  $P$ .

It is for some purposes convenient to use Fedorow coordinates.

Let planes through  $P$  perpendicular to the axes of  $x, y$ , and  $z$  meet these axes in  $A, B$ , and  $C$  respectively; then the lengths  $OA, OB, OC$ † are the *Fedorow coordinates* of the point  $P$ ; we shall denote

these coordinates by  $\xi, \eta, \zeta$ .

They can be readily expressed in terms of  $x, y, z$ ; for draw  $PN$  parallel to  $OZ$  to meet the plane  $AOB$  in  $N$ , and draw  $NL$  parallel to  $OY$ , meeting  $OX$  in  $L$  (Fig. 56).

\* Also called 'Cubic system.'

† With the usual convention of sign.

Then (since the axes make angles  $\alpha, \beta, \gamma$  with one another and  $PA$  is perpendicular to  $OX$ ) we have, projecting the sides of the polygon  $PNLA$  on  $OX$ ,  $LA = LN \cos \gamma + NP \cos \beta$ ; or since  $LA = \xi - x$ ,  $LN = y$ ,  $NP = z$ .

$$\text{Similarly } \left. \begin{aligned} \xi &= x + y \cos \gamma + z \cos \beta \\ \eta &= x \cos \gamma + y + z \cos \alpha \\ \zeta &= x \cos \beta + y \cos \alpha + z \end{aligned} \right\} \dots \dots \dots (i).$$

If the axes are mutually orthogonal  $\xi = x, \eta = y, \zeta = z$ ; i.e. the Fedorow coordinates are identical with the Cartesian.

Solving for  $x, y, z$  we have

$$\left. \begin{aligned} x &= \frac{1}{\Delta} [\xi \sin^2 \alpha + \eta (\cos \alpha \cdot \cos \beta - \cos \gamma) \\ &\quad + \zeta (\cos \gamma \cdot \cos \alpha - \cos \beta)] \\ y &= \frac{1}{\Delta} [\xi (\cos \alpha \cdot \cos \beta - \cos \gamma) + \eta \sin^2 \beta \\ &\quad + \zeta (\cos \beta \cdot \cos \gamma - \cos \alpha)] \\ z &= \frac{1}{\Delta} [\xi (\cos \gamma \cdot \cos \alpha - \cos \beta) \\ &\quad + \eta (\cos \beta \cdot \cos \gamma - \cos \alpha) + \zeta \sin^2 \gamma] \end{aligned} \right\} \dots (ii),$$

where  $\Delta \equiv (1 + 2 \cos \alpha \cdot \cos \beta \cdot \cos \gamma - \cos^2 \alpha - \cos^2 \beta - \cos^2 \gamma)$ .

§ 4. If we wish to find the coordinates of the point  $P$  referred to any other axes  $OX', OY', OZ'$  making angles  $\alpha', \beta', \gamma'$  with one another we proceed as follows.

Let  $x, y, z$  be the Cartesian coordinates of  $P$  referred to the old axes  $OX, OY, OZ$ ; let  $x', y', z'$  be the Cartesian coordinates referred to the new axes  $OX', OY', OZ'$ ; and let  $\xi, \eta, \zeta; \xi', \eta', \zeta'$  be the old and new Fedorow coordinates respectively.

Let the cosines of the angles which  $OX'$  makes with  $OX, OY, OZ$  be  $l_1, m_1, n_1$ \*; and let the corresponding quantities for  $OY'$  and  $OZ'$  be  $l_2, m_2, n_2$  and  $l_3, m_3, n_3$ .

Draw  $PA'$  perpendicular to  $OX'$ ; then  $OA' = \xi'$ .

Projecting the sides of the polygon  $OLNPA'$  (Fig. 56) on  $OX'$  we get

$$\text{similarly } \left. \begin{aligned} \xi' &= l_1 x + m_1 y + n_1 z \\ \eta' &= l_2 x + m_2 y + n_2 z \\ \zeta' &= l_3 x + m_3 y + n_3 z \end{aligned} \right\} \dots \dots \dots (iii).$$

\*  $l_1, m_1, n_1$  are connected by the relation

$$\Delta = l_1^2 \sin^2 \alpha + m_1^2 \sin^2 \beta + n_1^2 \sin^2 \gamma + 2 m_1 n_1 (\cos \beta \cdot \cos \gamma - \cos \alpha) + 2 n_1 l_1 (\cos \gamma \cdot \cos \alpha - \cos \beta) + 2 l_1 m_1 (\cos \alpha \cdot \cos \beta - \cos \gamma);$$

and so for  $l_2, m_2, n_2; l_3, m_3, n_3$ . This was proved in chap. ii, § 9 (p. 19).

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In Fig. 56 interchange the letters  $A$  and  $A'$ ,  $B$  and  $B'$ ,  $C$  and  $C'$ ; then projecting  $OLNPA$  on  $OA$  we get

$$\text{similarly } \left. \begin{aligned} \xi &= l_1 x' + l_2 y' + l_3 z' \\ \eta &= m_1 x' + m_2 y' + m_3 z' \\ \zeta &= n_1 x' + n_2 y' + n_3 z' \end{aligned} \right\} \dots \dots \dots (iv).$$

§ 5. If we have equations of the form

$$\begin{aligned} x' &= p_1 x + q_1 y + r_1 z \\ y' &= p_2 x + q_2 y + r_2 z \\ z' &= p_3 x + q_3 y + r_3 z, \end{aligned}$$

we call

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix}$$

'the determinant of the transformation  $(x'y'z'; xyz)$ .'

If we have also

$$\begin{aligned} x'' &= P_1 x' + Q_1 y' + R_1 z' \\ y'' &= P_2 x' + Q_2 y' + R_2 z' \\ z'' &= P_3 x' + Q_3 y' + R_3 z', \end{aligned}$$

then

$$\begin{aligned} x'' &= (p_1 P_1 + p_2 Q_1 + p_3 R_1)x + (q_1 P_1 + q_2 Q_1 + q_3 R_1)y \\ &\quad + (r_1 P_1 + r_2 Q_1 + r_3 R_1)z \\ y'' &= (p_1 P_2 + p_2 Q_2 + p_3 R_2)x + (q_1 P_2 + q_2 Q_2 + q_3 R_2)y \\ &\quad + (r_1 P_2 + r_2 Q_2 + r_3 R_2)z \\ z'' &= (p_1 P_3 + p_2 Q_3 + p_3 R_3)x + (q_1 P_3 + q_2 Q_3 + q_3 R_3)y \\ &\quad + (r_1 P_3 + r_2 Q_3 + r_3 R_3)z, \end{aligned}$$

from which we readily verify that the determinant of the transformation  $(x''y''z''; xyz)$  is the product of the determinants of the transformations

$$(x''y''z''; x'y'z') \text{ and } (x'y'z'; xyz).$$

The transformations  $(xyz; x'y'z')$ ,  $(x'y'z'; xyz)$  are said to be reciprocal to one another; for instance the two transformations given by equations (i) and (ii)\* are reciprocal.

Since the determinant of the transformation  $(xyz; xyz)$ ,

$$\text{i. e. } \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1,$$

therefore the product of the determinants of two reciprocal transformations = 1.

Thus for example since the determinant of  $(\xi\eta\zeta; xyz)$ —equations (i)—is  $\Delta$ , therefore the determinant of  $(xyz; \xi\eta\zeta)$ —equations (ii)—is  $\frac{1}{\Delta}$ .

§ 6. Substituting in equations (iii) the values of  $x, y, z$  given by equations (ii), we have

$$\xi' = \frac{1}{\Delta} [\xi \{l_1 \sin^2 \alpha + m_1 (\cos \alpha \cdot \cos \beta - \cos \gamma) + n_1 (\cos \gamma \cdot \cos \alpha - \cos \beta)\} \\ + \eta \{l_1 (\cos \alpha \cdot \cos \beta - \cos \gamma) + m_1 \sin^2 \beta + n_1 (\cos \beta \cdot \cos \gamma - \cos \alpha)\} \\ + \zeta \{l_1 (\cos \gamma \cdot \cos \alpha - \cos \beta) + m_1 (\cos \beta \cdot \cos \gamma - \cos \alpha) + n_1 \sin^2 \gamma\}]$$

and two similar equations . . . . . (v).

The determinant of this transformation  $(\xi'\eta'\zeta'; \xi\eta\zeta)$  is the product of the determinants of  $(\xi'\eta'\zeta'; xyz)$  and  $(xyz; \xi\eta\zeta)$ , and is therefore equal to  $\frac{D}{\Delta}$ , where  $D$  denotes the determinant

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}.$$

Again from (ii)

$$x' = \frac{1}{\Delta'} [\xi' \sin^2 \alpha' + \eta' (\cos \alpha' \cdot \cos \beta' - \cos \gamma') \\ + \zeta' (\cos \gamma' \cdot \cos \alpha' - \cos \beta')]$$

and two similar equations; where

$$\Delta' \equiv (1 + 2 \cos \alpha' \cdot \cos \beta' \cdot \cos \gamma' - \cos^2 \alpha' - \cos^2 \beta' - \cos^2 \gamma').$$

Substituting in (iv) we find

$$\xi = \frac{1}{\Delta} [\xi' \{l_1 \sin^2 \alpha' + l_2 (\cos \alpha' \cdot \cos \beta' - \cos \gamma') + l_3 (\cos \gamma' \cdot \cos \alpha' - \cos \beta')\} \\ + \eta' \{l_1 (\cos \alpha' \cdot \cos \beta' - \cos \gamma') + l_2 \sin^2 \beta' + l_3 (\cos \beta' \cdot \cos \gamma' - \cos \alpha')\} \\ + \zeta' \{l_1 (\cos \gamma' \cdot \cos \alpha' - \cos \beta') + l_2 (\cos \beta' \cdot \cos \gamma' - \cos \alpha') + l_3 \sin^2 \gamma'\}]$$

and two similar equations . . . . . (vi).

Hence the determinant of the transformation  $(\xi \eta \zeta; \xi' \eta' \zeta')$

$$= \frac{\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}}{\Delta'} = \frac{D}{\Delta'}.$$

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Another expression for  $\xi, \eta, \zeta$  in terms of  $\xi', \eta', \zeta'$  is found by solving for  $x, y, z$  from equations (iii) and substituting in (i).

We have then

$$\xi = \frac{1}{D} \left[ \xi' \begin{vmatrix} 1 & \cos \gamma \cos \beta \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} + \eta' \begin{vmatrix} l_1 & m_1 & n_1 \\ 1 & \cos \gamma \cos \beta \\ l_3 & m_3 & n_3 \end{vmatrix} + \zeta' \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ 1 & \cos \gamma \cos \beta \end{vmatrix} \right]$$

and two similar equations . . . . . (vii).

Again substituting from equations (iv) in (ii) we have

$$x = \frac{1}{\Delta} [x' \{l_1 \sin^2 a + m_1 (\cos a \cdot \cos \beta - \cos \gamma) + n_1 (\cos \gamma \cdot \cos a - \cos \beta)\} \\ + y' \{l_2 \sin^2 a + m_2 (\cos a \cdot \cos \beta - \cos \gamma) + n_2 (\cos \gamma \cdot \cos a - \cos \beta)\} \\ + z' \{l_3 \sin^2 a + m_3 (\cos a \cdot \cos \beta - \cos \gamma) + n_3 (\cos \gamma \cdot \cos a - \cos \beta)\}]$$

and two similar equations . . . . . (viii).

The determinant of this transformation ( $x y z; x' y' z'$ ) = the product of the determinants of ( $x y z; \xi \eta \zeta$ ) and ( $\xi \eta \zeta; x' y' z'$ ) =  $\frac{D}{\Delta}$ .

Similarly from the equations

$$x' = \frac{1}{\Delta} [\xi' \sin^2 a' + \eta' (\cos a' \cdot \cos \beta' - \cos \gamma') \\ + \zeta' (\cos \gamma' \cdot \cos a' - \cos \beta')], \text{ \&c.}$$

and (iii) we have

$$x' = \frac{1}{\Delta'} [x' \{l_1 \sin^2 a' + l_2 (\cos a' \cdot \cos \beta' - \cos \gamma') + l_3 (\cos \gamma' \cdot \cos a' - \cos \beta')\} \\ + y' \{m_1 \sin^2 a' + m_2 (\cos a' \cdot \cos \beta' - \cos \gamma') + m_3 (\cos \gamma' \cdot \cos a' - \cos \beta')\} \\ + z' \{n_1 \sin^2 a' + n_2 (\cos a' \cdot \cos \beta' - \cos \gamma') + n_3 (\cos \gamma' \cdot \cos a' - \cos \beta')\}]$$

and two similar equations . . . . . (ix).

The determinant of this transformation ( $x' y' z'; x y z$ ) is evidently  $\frac{D}{\Delta'}$ .

Another expression for  $x', y', z'$  in terms of  $x, y, z$  is found by solving for  $x', y', z'$  from (iv), and substituting for  $\xi, \eta, \zeta$  from (i).

We have then

$$x' = \frac{1}{D} \left[ x \begin{vmatrix} 1 & l_2 & l_3 \\ \cos \gamma & m_2 & m_3 \\ \cos \beta & n_2 & n_3 \end{vmatrix} + y \begin{vmatrix} \cos \gamma & l_2 & l_3 \\ 1 & m_2 & m_3 \\ \cos \alpha & n_2 & n_3 \end{vmatrix} + z \begin{vmatrix} \cos \beta & l_2 & l_3 \\ \cos \alpha & m_2 & m_3 \\ 1 & n_2 & n_3 \end{vmatrix} \right]$$

and two similar equations . . . . . (x).

It is interesting to compare equations (v) and (vi) with equations (viii) and (ix); for other forms of the equations (v), (vii), (viii), and (x) see note I at the end of this chapter.

Since the product of the determinants of the two transformations  $(x' y' z'; xyz)$  and  $(xyz; x' y' z') = 1$ ,

$$\therefore \frac{D}{\Delta} \cdot \frac{D}{\Delta} = 1 \text{ or } D^2 = \Delta \Delta' \dagger.$$

An interesting case is that for which the determinant of  $(x' y' z'; xyz)$ , and therefore the determinants of  $(xyz; x' y' z')$ ,  $(\xi' \eta' \zeta'; \xi \eta \zeta)$ ,  $(\xi \eta \zeta; \xi' \eta' \zeta') = \pm 1$ .

Then  $D^2 = \Delta'^2$ , and  $\therefore \Delta = \Delta'$ .

Since  $\Delta \equiv 1 - (\cos \gamma - \cos \alpha \cdot \cos \beta)^2 - \cos^2 \alpha \cdot \sin^2 \beta - \cos^2 \beta$ , we cannot have  $\Delta = \Delta' = 1$  unless

$$\cos \alpha = \cos \beta = \cos \gamma = \cos \alpha' = \cos \beta' = \cos \gamma' = 0 \ddagger,$$

i.e. both sets of axes are mutually orthogonal; if however  $\Delta = \Delta' \neq 1$  it does not necessarily follow that

$$\alpha' = \alpha, \beta' = \beta, \gamma' = \gamma,$$

though this is quite consistent with  $\Delta = \Delta'$ .

If  $\alpha' = \alpha, \beta' = \beta, \gamma' = \gamma$  the set of axes  $OX', OY', OZ'$  is either congruent or enantiomorphous to the set  $OX, OY, OZ$ .

§ 7. We can obtain a criterion as to the sign of the determinant of transformation in the general case by the following reasoning.

\* For neither  $\Delta$  nor  $\Delta'$  can be 0. For if e.g.

$$1 + 2 \cos \alpha \cdot \cos \beta \cdot \cos \gamma - \cos^2 \alpha - \cos^2 \beta - \cos^2 \gamma = 0,$$

solving for  $\cos \gamma$  we see that  $\pm \gamma = \alpha \pm \beta \pm$  a multiple of  $2\pi$ ; and therefore the lines  $OX, OY, OZ$  are coplanar contrary to hypothesis. The relation  $D^2 = \Delta \Delta'$  can also be proved by considering the product of the determinants of  $(\xi' \eta' \zeta'; \xi \eta \zeta)$  and  $(\xi \eta \zeta; \xi' \eta' \zeta')$ .

† This is the relation between the cosines of the angles which any six straight lines make with one another. See note II at the end of this chapter.

‡ The angles  $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$  are of course considered all real.

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Suppose the original axes  $OX, OY, OZ$  fixed, while the positions of the new axes  $OX', OY', OZ'$  are varied *continuously*; then the determinant of any one of the transformations of § 6 will vary also *continuously*, and therefore cannot change its sign without passing through the value 0 or  $\infty$ . This can only happen when  $\Delta' = 0^*$ , i.e. if  $OX', OY', OZ'$  become coplanar; for the determinant in question has the value  $(\frac{\Delta'}{\Delta})^{\pm \frac{1}{2}}$ . Hence if the positions of  $OX', OY', OZ'$  are varied continuously in such a way that they never become coplanar, the sign of  $\sqrt{\frac{\Delta'}{\Delta}}$  (and of  $\sqrt{\frac{\Delta}{\Delta'}}$ ) remains unaltered.

First suppose the set  $OX', OY', OZ'$  congruent to the set  $OX, OY, OZ$ ; then  $OX'$  can be brought to coincide with  $OX$ ,  $OY'$  with  $OY$ ,  $OZ'$  with  $OZ$  by a continuous variation of position of the kind considered. Therefore the sign of the determinant of transformation from  $OX, OY, OZ$  to  $OX', OY', OZ'$  is the same as that of the transformation from

$OX, OY, OZ$  to  $OX, OY, OZ$ , i.e. the same as that of  $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ ,

and the determinant is positive. Now in the case considered  $\Delta' = \Delta$ , and therefore the determinant of transformation from a set  $OX, OY, OZ$  to a *congruent* set  $OX', OY', OZ' = +1$ .

Similarly if the set  $OX', OY', OZ'$  is enantiomorphous to the set  $OX, OY, OZ$ ;  $OX'$  can be brought into coincidence with  $OX$ ,  $OY'$  with  $OZ$ ,  $OZ'$  with  $OY$  by a continuous change of position of the kind considered. Hence the sign of the determinant of transformation from  $OX, OY, OZ$  to  $OX', OY',$

$OZ'$  is that of  $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$ , and the determinant is negative.

Also again  $\Delta' = \Delta$ , and therefore the determinant of transformation from a set  $OX, OY, OZ$  to an *enantiomorphous* set  $OX', OY', OZ' = -1$ .

Now let  $OX', OY', OZ'$  have any non-coplanar positions.

In the plane  $X'OY'$  draw  $OE'$  perpendicular to  $OX'$ , and on the same side of  $OX'$  (produced) as  $OY'$ ; similarly draw  $OF'$  perpendicular to  $OX'$  in the plane  $X'OZ'$  on the same side of  $OX'$  as  $OZ'$ . Let  $OE, OF$  have the same positions with regard to  $OX, OY, OZ$  as  $OE', OF'$  have with regard to  $OX', OY', OZ'$ .

Now let  $OY', OZ'$  be moved in the planes  $X'OY', X'OZ'$  respectively (so that they remain on the same side of  $OX'$

\* Assuming  $\Delta \neq 0$ ; i. e. assuming  $OX, OY, OZ$  not coplanar.

as  $OE', OF'$  until the angle  $X'OY' = XOY$  and the angle  $X'OZ' = XOZ$ .

Now let  $OE', OF'$  be rotated together about  $OX'$  so that  $OE', OF'$  never make an angle of 0 or  $\pi$  with one another until the angle  $E'OF'$  measured in a positive direction equals  $EOF$  or (if that is impossible)  $2\pi - EOF$  (measuring  $EOF$  in the same direction). Then in the former case the determinant of transformation from  $OX, OY, OZ$  to the altered positions of  $OX', OY', OZ'$  is +1, and therefore the determinant of transformation from  $OX, OY, OZ$  to the original positions of  $OX', OY', OZ'$  is positive; similarly in the latter case (when the altered value of  $E'OF' = 2\pi - EOF$ ) the determinant of transformation from  $OX, OY, OZ$  to the altered positions of  $OX', OY', OZ'$  is -1, and therefore the determinant of transformation from  $OX, OY, OZ$  to the original positions of  $OX', OY', OZ'$  is negative.

§ 8. Now let the set  $OX', OY', OZ'$  be obtained from any congruent set  $OX, OY, OZ$  by rotating the latter through an angle  $\theta$  about an axis  $ON$ .

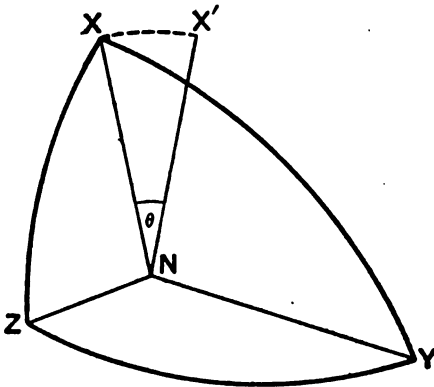


Fig. 57.

Let a sphere with centre  $O$  meet  $OX, OY, OZ, OX', OY', OZ', ON$  in  $X, Y, Z, X', Y', Z', N$  respectively; let the arcs  $NX = NX' = \lambda$ ,  $NY = NY' = \mu$ ,  $NZ = NZ' = \nu$ , and let

$$\cos \lambda = l, \cos \mu = m, \cos \nu = n.$$

Then we have

$$YZ = a, ZX = \beta, XY = \gamma, X'NX = \theta, \\ \cos XX' = l_1, \cos YX' = m_1, \cos ZX' = n_1$$

(Fig. 57; the notation of § 8 is used).

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Then  $\cos ZNY = \frac{\cos a - \cos \mu \cdot \cos \nu}{\sin \mu \cdot \sin \nu}$ , &c.

and therefore

$$\sin ZNY = + \frac{\sqrt{\sin^2 a - m^2 - n^2 + 2 \cos a \cdot mn}}{\sin \mu \cdot \sin \nu}, \text{ \&c.}$$

From the spherical triangle  $XNX'$ ,

$$l_1 = \cos^2 \lambda + \sin^2 \lambda \cdot \cos \theta = l^2 (1 - \cos \theta) + \cos \theta.$$

From the triangle  $X'NY$

$$m_1 = \cos \lambda \cdot \cos \mu + \sin \lambda \cdot \sin \mu \cdot \cos (YNX - \theta) = lm (1 - \cos \theta) + \cos \theta \cdot \cos \gamma + \sin \theta \sqrt{\sin^2 \gamma - l^2 - m^2 + 2 \cos \gamma \cdot lm}.$$

From the triangle  $X'NZ$

$$n_1 = \cos \lambda \cdot \cos \nu + \sin \lambda \cdot \sin \nu \cdot \cos (XNZ + \theta) = ln (1 - \cos \theta) + \cos \theta \cdot \cos \beta - \sin \theta \sqrt{\sin^2 \beta - l^2 - n^2 + 2 \cos \beta \cdot ln}.$$

We have similar expressions for  $l_2, m_2, n_2$ ;  $l_3, m_3, n_3$ .

Hence if we are given the coordinates of any point referred to any set of axes, we can obtain the coordinates of the point referred to the set obtained by rotating the original set through a known angle about a given axis. For all we require is the cosines of the angles which the new axes make with the old (§ 6), and these we have now shown how to obtain.

Now let  $OX, OY, OZ$  be brought to  $OX', OY', OZ'$  by some known operation of the second sort.

Produce  $XO, YO, ZO$  to  $X'', Y'', Z''$  respectively; then  $OX, OY, OZ$  are brought into coincidence with  $OX'', OY'', OZ''$  by an inversion about  $O$ , and  $OX'', OY'', OZ''$  are brought to  $OX', OY', OZ'$  by some known operation of the first sort, i. e. by some known rotation. The above investigation gives us the cosines of the angles which  $OX', OY', OZ'$  make with  $OX'', OY'', OZ''$ , and by multiplying these by  $-1$  we obtain the cosines of the angles which  $OX', OY', OZ'$  make with  $OX, OY, OZ$ . Hence we obtain the coordinates of any point referred to  $OX', OY', OZ'$  in terms of the coordinates referred to  $OX, OY, OZ$ .

§ 9. A useful case is that in which the set  $OX', OY', OZ'$  is the reflexion of the set  $OX, OY, OZ$  in a plane the normal to which makes angles with  $OX, OY, OZ$  whose cosines are  $l, m, n$  respectively.

In this case putting  $\theta = \pi$  in the equations of § 8 and

changing the signs of the right-hand sides of those equations, we have

$$\left. \begin{aligned} l_1 &= 1 - 2l^2, & m_1 &= \cos \gamma - 2lm, & n_1 &= \cos \beta - 2nl \\ l_2 &= \cos \gamma - 2lm, & m_2 &= 1 - 2m^2, & n_2 &= \cos \alpha - 2mn \\ l_3 &= \cos \beta - 2nl, & m_3 &= \cos \alpha - 2mn, & n_3 &= 1 - 2n^2. \end{aligned} \right\}.$$

§ 10. *To find the coordinates of all the points equivalent to any point  $P$  in any group of movements.*

Suppose we take any fixed reference-axes  $OX, OY, OZ$ , and suppose the coordinates of the point  $P$  with reference to these axes to be known.

Let the operation  $L$  of the group bring  $P$  to coincide with the equivalent point  $P'$ .

Then if  $L$  brings the reference-axes into the positions  $OX', OY', OZ'$ , the coordinates of the point  $P'$  referred to  $OX', OY', OZ'$  are evidently the same as the known coordinates of  $P$  referred to  $OX, OY, OZ$ ; and we have shown in the previous sections how to express the coordinates of  $P'$  referred to  $OX, OY, OZ$  in terms of its coordinates referred to  $OX', OY', OZ'$ , when  $OX, OY, OZ$  are brought into coincidence with  $OX', OY', OZ'$  by any operation of the first or second sort.

If, however, we choose our reference-axes in each case as conveniently as possible, we can almost always write down the coordinates of a system of equivalent points by inspection.

Take, for instance, the case of  $D_{2d}$  whose operations are 1,  $A$ ;  $B$ ,  $A.B$ ;  $S_d$ ,  $A.S_d$ ;  $B.S_d$ ,  $A.B.S_d$  (p. 69). Taking the axes of  $z, y$  as the axes of  $A$  and  $B$  respectively, and the plane  $x = y$  as the plane of  $S_d$ ; we see that the point  $(xyz)$  is brought to coincide with  $(\bar{x}\bar{y}z)$  by  $A$  and with  $(\bar{x}y\bar{z})$  by  $B$ ; similarly  $(\bar{x}\bar{y}z)$  is brought to  $(xyz)$  by  $B$ , and therefore  $(xyz)$  is brought to  $(x\bar{y}\bar{z})$  by  $A.B$ .

Again  $(xyz)$ ,  $(\bar{x}\bar{y}z)$ ,  $(\bar{x}y\bar{z})$ ,  $(x\bar{y}\bar{z})$  are brought respectively to  $(y\bar{x}z)$ ,  $(\bar{y}\bar{x}z)$ ,  $(y\bar{x}\bar{z})$ ,  $(\bar{y}x\bar{z})$  by  $S_d$ , and therefore  $(xyz)$  is brought to  $(y\bar{x}z)$ ,  $(\bar{y}\bar{x}z)$ ,  $(y\bar{x}\bar{z})$ ,  $(\bar{y}x\bar{z})$  by  $S_d$ ,  $A.S_d$ ,  $B.S_d$ ,  $A.B.S_d$ . The system of equivalent points of  $D_{2d}$  is therefore that given in § 14.

In the following sections we shall point out which reference-axes are the most suitable, and leave the reader to verify the conclusions stated.

§ 11. In the case of the groups  $C_1$ ,  $C_2$  belonging to the triclinic system (§ 2), there is no line specially pointed out by the symmetry of the groups as a proper line to be taken as an axis of coordinates. We take therefore *any* three lines as

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axes (in general no two of these lines are at right angles to one another—hence the name ‘triclinic’); let  $xyz$  be the Cartesian coordinates of any point  $P$  and let  $\xi\eta\zeta$  be its Fedorow coordinates, then the system of points equivalent to (and including)  $P$  is

$$\begin{array}{ll} \text{for} & C_1 \dots \dots \dots xyz. \quad \xi\eta\zeta \\ \text{and for} & C_2 \dots xyz, \bar{x}\bar{y}\bar{z}. \quad \xi\eta\zeta, -\xi^{-}\eta^{-}\zeta^{-}. \end{array}$$

§ 12. In the case of  $C_2, C_2, C_{2h}$  belonging to the monoclinic system, the line perpendicular to the symmetry-plane, or parallel to the 2-al rotation-axis is pointed out by the symmetry of the groups as a suitable line for one of the coordinate axes. Take then this line as the axis of  $z$ , and *any* two lines perpendicular to it (not in general at right angles to each other—hence the name ‘monoclinic’) as axes of  $x$  and  $y$ .

Then a system of equivalent points is

$$\begin{array}{ll} \text{for} & C_2 \dots xyz, xy\bar{z}. \quad \xi\eta\zeta, \xi\eta^{-}\zeta. \\ \text{for} & C_2 \dots xyz, \bar{x}\bar{y}z. \quad \xi\eta\zeta, -\xi^{-}\eta\zeta. \\ \text{and for } C_{2h} \dots & xyz, \bar{x}\bar{y}z, xy\bar{z}, \bar{x}\bar{y}\bar{z}. \quad \xi\eta\zeta, -\xi^{-}\eta\zeta, \xi\eta^{-}\zeta, -\xi^{-}\eta^{-}\zeta. \end{array}$$

§ 13. For the group  $C_{2v}$  we take the 2-al rotation-axis as axis of  $z$ , and the two lines perpendicular to it and lying in the symmetry-planes as axes of  $x$  and  $y$ ; for the other two groups of the orthorhombic system,  $D_2$  and  $D_{2h}$ , we take the three 2-al symmetry-axes as coordinate axes.

In each case the three coordinate axes are mutually orthogonal; the Cartesian and Fedorow coordinates are therefore identical. We have as the systems of Cartesian coordinates

$$\begin{array}{ll} \text{for} & C_{2v} \dots xyz, \bar{x}\bar{y}z, xy\bar{z}, \bar{x}\bar{y}\bar{z}. \\ \text{for} & D_2 \dots xyz, \bar{x}\bar{y}z, \bar{x}y\bar{z}, x\bar{y}\bar{z}. \\ \text{for} & D_{2h} \dots xyz, \bar{x}\bar{y}z, \bar{x}y\bar{z}, xy\bar{z}; \{ \\ & \quad xy\bar{z}, \bar{x}\bar{y}\bar{z}, \bar{x}y\bar{z}, x\bar{y}\bar{z}. \} \end{array}$$

and the Fedorow coordinates are identical with these. We notice that every symmetry-operation of any group of the triclinic, monoclinic, or orthorhombic systems brings each reference-axis into coincidence with itself.

§ 14. In the tetragonal system we again take the three coordinate axes as mutually orthogonal.

The  $z$  axis is the 4-al symmetry-axis of the second sort for  $C_4'$  and  $D_{2d}$ , and the 4-al rotation-axis for  $C_4, C_{4h}, C_{4v}, D_4, D_{4h}$ .

The axes of  $x$  and  $y$  are 2-al rotation-axes for  $D_{2d}, D_4$ , and  $D_{4h}$ , lie in symmetry-planes for  $C_{4v}$ , and are any two lines

\*  $\bar{x}$  is printed for  $-x$ ,  $-\xi$  for  $-\xi$ , &c.  $-\xi$  would be written  $\bar{\xi}$ , &c.

perpendicular to each other and to the  $z$  axis in the case of  $C_4$ ,  $C_4'$ , and  $C_{4h}$ .

The systems of Cartesian coordinates are

for	$C_4'$	$\dots xyz, \bar{y}az, \bar{x}\bar{y}z, y\bar{x}\bar{z}.$	
for	$D_{2d}$	$\dots xyz, \bar{x}\bar{y}z, \bar{x}y\bar{z}, a\bar{y}\bar{z};$	$yaz, \bar{y}\bar{x}z, y\bar{x}\bar{z}, \bar{y}az.$
for	$C_4$	$\dots xyz, \bar{y}az, \bar{x}\bar{y}z, y\bar{x}\bar{z}.$	
for	$C_{4h}$	$\dots xyz, \bar{y}az, \bar{x}\bar{y}z, y\bar{x}\bar{z};$	$ay\bar{z}, \bar{y}az, \bar{x}\bar{y}\bar{z}, y\bar{x}\bar{z}.$
for	$C_{4v}$	$\dots xyz, \bar{y}az, \bar{x}\bar{y}z, y\bar{x}\bar{z};$	$yaz, a\bar{y}z, \bar{y}\bar{x}z, \bar{x}yz.$
for	$D_4$	$\dots xyz, \bar{y}az, \bar{x}\bar{y}z, y\bar{x}\bar{z};$	$\bar{x}y\bar{z}, yaz, a\bar{y}\bar{z}, \bar{y}\bar{x}\bar{z}.$
for	$D_{4h}$	$\dots xyz, \bar{y}az, \bar{x}\bar{y}z, y\bar{x}\bar{z};$	$\bar{x}y\bar{z}, yaz, a\bar{y}\bar{z}, \bar{y}\bar{x}\bar{z};$
		$ay\bar{z}, \bar{y}az, \bar{x}\bar{y}\bar{z}, y\bar{x}\bar{z};$	$\bar{x}yz, yaz, a\bar{y}z, \bar{y}\bar{x}z.\}$

The Fedorow coordinates are identical with these.

§ 15. In the hexagonal system there are various methods of choosing the coordinate axes.

(1) Sometimes they are taken mutually orthogonal as in the orthorhombic and tetragonal systems, we shall not however discuss this method.

(2) A method often employed in the case of the first subdivision of the hexagonal system is the following:—As the axis of  $z$  is chosen, for the groups  $C_3$  and  $C_{3i}$ , any line, for the groups  $C_{3v}$  and  $D_{3d}$  any line lying in a symmetry-plane, and for the group  $D_3$  (and  $D_{3d}$ ) any line lying in a plane perpendicular to a 2-al rotation-axis. For the axes of  $x$  and  $y$  are taken the positions into which the axis of  $z$  is brought by rotations through  $\frac{2\pi}{3}$  and  $\frac{4\pi}{3}$  respectively about the 3-al rotation-axis, so that the three coordinate axes are situated symmetrically with respect to the 3-al axis.

Systems of equivalent points are then

for	$C_3$	$\dots xyz, xzy, yzx.$	
for	$C_{3i}$	$\dots xyz, xzy, yzx; \bar{x}\bar{y}\bar{z}, \bar{z}\bar{x}\bar{y}, \bar{y}\bar{z}\bar{x}.$	
for	$C_{3v}$	$\dots xyz, xzy, yzx; yax, xzy, zyx.$	
for	$D_3$	$\dots xyz, xzy, yzx; \bar{y}\bar{x}\bar{z}, \bar{x}\bar{z}\bar{y}, \bar{z}\bar{y}\bar{x}.$	
for	$D_{3d}$	$\dots xyz, xzy, yzx; \bar{y}\bar{x}\bar{z}, \bar{x}\bar{z}\bar{y}, \bar{z}\bar{y}\bar{x};$	$yxz, xzy, zyx; \bar{x}\bar{y}\bar{z}, \bar{z}\bar{x}\bar{y}, \bar{y}\bar{z}\bar{x}.\}$

We can obtain the systems of Fedorow coordinates by putting  $\xi, \eta, \zeta$  for  $x, y, z$  in the above.

This method is, however, not so suitable for the second and third subdivisions of the system, and in what follows we shall use method

(3) In this case we refer any point to four coordinate axes,  $OX_1, OX_2, OX_3$ , and  $OZ$ . The axis of  $z$  is in each case the 3-al

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or 6-al rotation-axis, and the three remaining axes make angles of  $\frac{2\pi}{3}$  with one another and are all perpendicular to the axis of  $z$ .

We denote the Fedorow coordinates of any point  $P$  by  $\xi_1, \xi_2, \xi_3, \zeta$ ; the ordinary definition of Cartesian coordinates is not applicable to the case of more than three reference-axes.

The quantities  $\xi_1, \xi_2, \xi_3, \zeta$  must be connected by one relation, we may find it as follows:—

Draw perpendiculars  $PE_1, PE_2, PE_3$  to  $OX_1, OX_2, OX_3$ , and  $PP'$  perpendicular to the plane of these three axes (Fig. 58); then  $P'E_1$  is perpendicular to  $OE_1$ ,  $P'E_2$  to  $OE_2$ , and  $P'E_3$  to  $OE_3$ .

Now the angle

$$P'OX_2 - P'OX_1 = \frac{2\pi}{3},$$

and

$$P'OX_2 + P'OX_3 = \frac{4\pi}{3};$$

adding these two equations we have  $P'OX_2$

$$= \pi - \left( \frac{P'OX_1 - P'OX_3}{2} \right),$$

subtracting we have

$$\left( \frac{P'OX_1 + P'OX_3}{2} \right) = \frac{\pi}{3}.$$

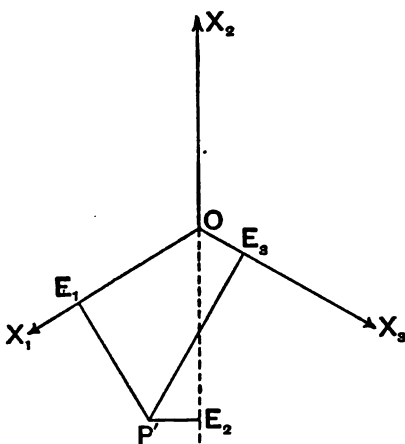


Fig. 58.

$$\therefore OP' (\cos P'OX_1 + \cos P'OX_2 + \cos P'OX_3)$$

$$= OP' \left[ 2 \cos \left( \frac{P'OX_1 + P'OX_3}{2} \right) \cos \left( \frac{P'OX_1 - P'OX_3}{2} \right) \right.$$

$$\left. + \cos P'OX_2 \right] = OP' \left( -2 \cos \frac{\pi}{3} \cdot \cos P'OX_2 + \cos P'OX_2 \right) = 0,$$

and therefore  $OE_1 + OE_2 + OE_3 = 0$ .

Hence

$$\xi_1 + \xi_2 + \xi_3 = 0, \text{ for } OE_1 = \xi_1, OE_2 = \xi_2, OE_3 = \xi_3.$$

As the position of  $OX_1$  we take, for  $C_3, C_{3i}, C_{3h}, C_6, C_{6h}$  any line perpendicular to the symmetry-axis, for  $C_{3v}, C_{6v}$  a line in a symmetry-plane perpendicular to the rotation-axis, and for  $D_3, D_{3d}, D_{3h}, D_6, D_{6h}$  a 2-al symmetry-axis.

The Fedorow coordinates of an equivalent system are then

for $D_{6h}$	$\xi_1 \xi_2 \xi_3 \zeta$	$\xi_3 \xi_1 \xi_2 \zeta$	$\xi_2 \xi_3 \xi_1 \zeta$	I.
	$-\xi_1 - \xi_2 - \xi_3 \zeta$	$-\xi_3 - \xi_1 - \xi_2 - \zeta$	$-\xi_2 - \xi_3 - \xi_1 \zeta$	II.
	$\xi_3 \xi_2 \xi_1 - \zeta$	$\xi_2 \xi_1 \xi_3 - \zeta$	$\xi_1 \xi_3 \xi_2 - \zeta$	III.
	$-\xi_3 - \xi_2 - \xi_1 - \zeta$	$-\xi_2 - \xi_1 - \xi_3 - \zeta$	$-\xi_1 - \xi_3 - \xi_2 - \zeta$	IV.
	$\xi_1 \xi_2 \xi_3 - \zeta$	$\xi_3 \xi_1 \xi_2 - \zeta$	$\xi_2 \xi_3 \xi_1 - \zeta$	V.
	$-\xi_1 - \xi_2 - \xi_3 - \zeta$	$-\xi_3 - \xi_1 - \xi_2 - \zeta$	$-\xi_2 - \xi_3 - \xi_1 - \zeta$	VI.
	$\xi_3 \xi_2 \xi_1 \zeta$	$\xi_2 \xi_1 \xi_3 \zeta$	$\xi_1 \xi_3 \xi_2 \zeta$	VII.
	$-\xi_3 - \xi_2 - \xi_1 \zeta$	$-\xi_2 - \xi_1 - \xi_3 \zeta$	$-\xi_1 - \xi_3 - \xi_2 \zeta$	VIII.

- for  $C_3$  the 1st of the above lines,  
 for  $C_{3d}$  the 1st and 6th of the above lines,  
 for  $C_{3v}$  the 1st and 7th of the above lines,  
 for  $D_3$  the 1st and 3rd of the above lines,  
 for  $D_{3d}$  the 1st, 3rd, 6th and 8th of the above lines,  
 for  $C_{3h}$  the 1st and 5th of the above lines,  
 for  $D_{3h}$  the 1st, 3rd, 5th and 7th of the above lines,  
 for  $C_6$  the 1st and 2nd of the above lines,  
 for  $C_{6h}$  the 1st, 2nd, 5th and 6th of the above lines,  
 for  $C_{6v}$  the 1st, 2nd, 7th and 8th of the above lines,  
 for  $D_6$  the 1st, 2nd, 3rd and 4th of the above lines.

In each case the relation  $\xi_1 + \xi_2 + \xi_3 = 0$  holds.

We notice that every symmetry-operation of any group of the tetragonal or hexagonal systems brings the axis of  $z$  into self-coincidence; and brings any one of the other reference-axes into coincidence with itself or some other reference-axis perpendicular to the axis of  $z$ .

§ 16. For the regular system we take the three 2-al symmetry-axes of  $T$ ,  $T_h$ , and  $T_d$ ; or the three 4-al symmetry-axes of  $O$  and  $O_h$  as axes of reference. They are mutually perpendicular so that the Fedorow coordinates coincide with the Cartesian.

We have for  $O_h$  the system

$xyz, x\bar{y}\bar{z}, \bar{x}y\bar{z}, \bar{x}\bar{y}z; zxy, \bar{z}a\bar{y}, \bar{z}\bar{x}y, z\bar{x}\bar{y}; yza, \bar{y}\bar{z}a, y\bar{z}\bar{a}, \bar{y}z\bar{a}$	I.
$ya\bar{z}, \bar{y}xz, y\bar{x}z, \bar{y}\bar{x}\bar{z}; az\bar{y}, a\bar{z}y, \bar{a}z\bar{y}, \bar{a}\bar{z}y; zy\bar{x}, \bar{z}\bar{y}\bar{x}, \bar{z}ya, z\bar{y}a$	II.
$ay\bar{z}, a\bar{y}z, \bar{a}yz, \bar{a}\bar{y}\bar{z}; za\bar{y}, \bar{z}ay, \bar{z}\bar{x}\bar{y}, z\bar{x}y; yz\bar{x}, \bar{y}\bar{z}\bar{x}, y\bar{z}a, \bar{y}za$	III.
$ya\bar{z}, \bar{y}a\bar{z}, y\bar{x}\bar{z}, \bar{y}\bar{x}\bar{z}; azy, a\bar{z}\bar{y}, \bar{a}zy, \bar{a}\bar{z}\bar{y}; zy\bar{x}, \bar{z}\bar{y}\bar{x}, \bar{z}y\bar{x}, z\bar{y}\bar{x}$	IV.

- for  $T$  the 1st of the above lines,  
 for  $T_h$  the 1st and 3rd of the above lines,  
 for  $T_d$  the 1st and 4th of the above lines,  
 for  $O$  the 1st and 2nd of the above lines.

We notice that every symmetry-operation of any group of the regular system brings any reference-axis into coincidence with itself or one of the other reference-axes.

## NOTE I.

Some of the equations of § 6 may be put into a simpler form by using the *direction-ratios*  $L_1, M_1, N_1; L_2, M_2, N_2; L_3, M_3, N_3$ , of  $OX', OY', OZ'$  referred to the axes  $OX, OY, OZ$ , instead of the cosines of the angles which  $OX', OY', OZ'$  make with these axes. We have

$$\begin{aligned} l_1 &= L_1 + M_1 \cos \gamma + N_1 \cos \beta \\ m_1 &= L_1 \cos \gamma + M_1 + N_1 \cos \alpha \\ n_1 &= L_1 \cos \beta + M_1 \cos \alpha + N_1 \end{aligned}$$

and two similar sets of equations, by a method similar to that used in obtaining equations (i) of § 3.

Substituting for  $l_1, m_1, n_1$ , &c., in equations (v) and (viii), we have

$$\xi' = \xi L_1 + \eta M_1 + \zeta N_1, \text{ and two similar equations } \quad (\text{v})',$$

and

$$x = x' L_1 + y' L_2 + z' L_3 \text{ and two similar equations } \quad (\text{viii})'.$$

(v)' can also be proved by direct application of (iii).

Solving, we have

$$\xi = \frac{1}{\begin{vmatrix} L_1 & M_1 & N_1 \\ L_2 & M_2 & N_2 \\ L_3 & M_3 & N_3 \end{vmatrix}} \left[ \xi' \begin{vmatrix} M_2 & N_2 \\ M_3 & N_3 \end{vmatrix} + \eta' \begin{vmatrix} M_3 & N_3 \\ M_1 & N_1 \end{vmatrix} + \zeta' \begin{vmatrix} M_1 & N_1 \\ M_2 & N_2 \end{vmatrix} \right]$$

and two similar equations . . . . . (vii)';

$$x' = \frac{1}{\begin{vmatrix} L_1 & M_1 & N_1 \\ L_2 & M_2 & N_2 \\ L_3 & M_3 & N_3 \end{vmatrix}} \left[ x \begin{vmatrix} M_2 & M_3 \\ N_2 & N_3 \end{vmatrix} + y \begin{vmatrix} N_2 & N_3 \\ L_2 & L_3 \end{vmatrix} + z \begin{vmatrix} L_2 & L_3 \\ M_2 & M_3 \end{vmatrix} \right]$$

and two similar equations . . . . . (x).

The determinant of the transformation

$$(\xi' \eta' \zeta'; \xi \eta \zeta) \text{ or } (xyz; x' y' z') = \frac{D}{\Delta} = \begin{vmatrix} L_1 & M_1 & N_1 \\ L_2 & M_2 & N_2 \\ L_3 & M_3 & N_3 \end{vmatrix}.$$

## NOTE II.

The formula  $D^2 = \Delta \Delta'$  of § 6 of this chapter, and the formula at the end of § 8 of chapter ii, are particular cases of the following:—

If the cosine of the angle between  $b_i$  and  $b_j$ —any two of

the  $n$  straight lines  $b_1, b_2, b_3, \dots, b_{n-1}, b_n$ —be denoted by  $a_{ij}$  or  $\alpha_{ij}$  (with a convention as to the positive direction of the straight lines), then

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdot & \cdot & \cdot & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdot & \cdot & \cdot & a_{3n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & a_{m3} & \cdot & \cdot & \cdot & a_{mn} \end{vmatrix} = 0; \text{ if } n \text{ is } > 3.$$

Take any point  $A_2$  and draw a line through it in the direction of  $b_2$ ; take any point  $A_3$  on this line, and draw through it a line in the direction of  $b_3$ ; on this take any point  $A_4$  and draw a line through it in the direction of  $b_4$  and so on, till we get to a line through a point  $A_{n-1}$  parallel to  $b_{n-1}$ .

Through  $A_2$  draw a plane parallel to  $b_1$  and  $b_n$  meeting this line in  $A_n$ ; and through  $A_n$  draw a line (lying in this plane) in the direction of  $b_n$ , meeting a line through  $A_2$  in the opposite direction to  $b_1$  in the point  $A_1$ .

Then we have a closed polygon  $A_1 A_2 A_3 \dots A_{n-1} A_n$  whose sides are in the directions of  $b_1, b_2, b_3, \dots b_n$ ; let the length of its sides be respectively  $x_1, x_2, x_3, \dots x_n$ . Projecting all the sides of this polygon on each side in turn we get the equations

$$\begin{array}{ccccccccccc} x_1 a_{11} + x_2 a_{12} + x_3 a_{13} + & \dots & + x_n a_{1n} & = & 0. \\ x_1 a_{21} + x_2 a_{22} + x_3 a_{23} + & \dots & + x_n a_{2n} & = & 0. \\ x_1 a_{31} + x_2 a_{32} + x_3 a_{33} + & \dots & + x_n a_{3n} & = & 0. \\ & \dots & & & \dots & & & & \dots & & \\ x_1 a_{n1} + x_2 a_{n2} + x_3 a_{n3} + & \dots & + x_n a_{nn} & = & 0. \end{array}$$

Eliminating  $x_1, x_2, x_3, \dots, x_n$  we have the relation above stated.

Of course  $a_{11}, a_{22}, a_{33}, \dots a_{nn}$  all = +1 or -1 according to whether our conventions demand that a line should be considered as making an angle 0 or  $\pi$  with itself.

## CHAPTER VIII

## CRYSTAL FORMS.

§ 1. Those crystals which are brought to self-coincidence by the operations of a group  $G$  (which must be one of the thirty-two groups discussed in chapters v and vi) and by no other operations, are said to form a *class*. This class may be denoted by the same symbol as the corresponding group. If we take those planes of the polyhedron representing a crystal (p. 36) which are equidistant from the centre  $O$ , and which are therefore parallel to faces having the same physical properties, and produce them till they meet again, the polyhedron \*  $F$  so obtained is called a *form*. The feet of the perpendiculars from  $O$  on the faces of  $F$  are evidently a system of equivalent points; therefore the number of faces of any 'form' is *in general* equal to the number of operations in the corresponding group. As figures of various forms, both general and special, for each of the thirty-two crystal classes are given in nearly every textbook on the subject, we shall not reproduce them here, but shall content ourselves with proving a few general properties of such polyhedra.

§ 2. *If a crystal has a centre of symmetry,  $F$  has its faces parallel in pairs.*

For a plane is brought by inversion about a point  $O$  to coincidence with a parallel plane on the opposite side of and equidistant from  $O$ .

*A symmetry-plane cannot meet a face of  $F$  except along one of its edges.*

For if a face  $p$  of  $F$  meets a symmetry-plane in the line  $\Sigma$ ; then  $p'$ , the reflexion of  $p$  in the symmetry-plane, also passes through  $\Sigma$  and is a face of  $F$ ; so that  $\Sigma$  is the intersection of two faces of  $F$ , and is therefore an edge of  $F$  or  $p$ .

\* This polyhedron is not closed in the classes  $C_1, C_2, C_3, C_4, C_6, C_8, C_{12}, C_{24}, C_1, C_1, C_2, C_3, C_4, C_6, C_8, C_{12}, C_{24}$ , but is closed in the other twenty-one classes.

*An  $n$ -al rotation-axis meets  $F$  in one of its solid angles, at which  $2n$  or  $n$  edges of  $F$  intersect, according as the  $n$ -al axis does or does not lie in a symmetry-plane.*

For if a face  $p$  of  $F$  meets the axis in  $A$ , then the positions into which  $p$  is brought by successive rotations through  $\frac{2\pi}{n}$  about  $OA$  are also faces of  $F$  and pass through  $A$ ; these  $n$  faces meet in  $n$  lines which are edges of  $F$  if there is no other symmetry-operation of the group which leaves  $A$  unmoved, i. e. if there is no symmetry-plane passing through  $OA$ . If however there is a symmetry-plane through  $OA$ , there are  $n$  other symmetry-operations which leave  $OA$  unmoved (cf. p. 49); hence there will be  $2n$  faces of  $F$  passing through  $A$ , and therefore  $2n$  edges of  $F$  passing through  $A$ . If  $n = 2$ , and the axis does not lie in a symmetry-plane, we have an exception; for two planes through a point do not meet in two lines but in one. Hence a 2-al rotation-axis not lying in a symmetry-plane does not pass through an angle of  $F$ , but meets an edge of  $F$ . Since a rotation through  $\pi$  about the axis brings this edge into self-coincidence, the axis must be perpendicular to the edge and pass through its middle point.

§ 3. Since any symmetry-operation of  $F$  brings each system of equivalent symmetry-elements of  $F$  into self-coincidence, therefore every face of  $F$  must be situated with regard to each equivalent set of symmetry-elements similarly to all the other faces.

*Hence each face must meet one and only one axis of each set of equivalent rotation-axes.*

For if any face met none of the set, then none of the faces could meet any axis of the set; let then each face meet  $\lambda$  axes of the set, which we will suppose contains a equivalent  $n$ -al rotation-axes. Let  $N$  be the number of the faces of  $F$ . Then since  $2n$  or  $n$  faces meet in a point on each of the equivalent axes, according as the axes do or do not lie on symmetry-planes, therefore  $a = \frac{\lambda N}{2n}$  or  $\frac{\lambda N}{n}$  according as the axes do or do not lie on symmetry-planes. But (p. 50)  $a = \frac{N}{2n}$  or  $\frac{N}{n}$  according as the axes do or do not lie on symmetry-planes, and therefore in either case  $\lambda = 1$ .

§ 4. It may be easily verified (by considering the coordinates of systems of equivalent points or otherwise) that all the

groups in any one of the systems of § 2 of the previous chapter (p. 73) are subgroups of that group of the system which has the highest symmetry \*; e. g.

$$C_3, C_{3i}, C_{3v}, D_3, D_{3d}, C_{3h}, D_{3h}, C_6, C_{6h}, C_{6v}, D_6,$$

are subgroups of  $D_{6h}$ . Now the number of operations of a subgroup is an aliquot part of the number of operations of the group itself (p. 47); hence to obtain a polyhedron which is brought to self-coincidence by the operations of any group  $G_1$ , we take a certain fraction of the faces of a polyhedron whose symmetry-operations form that group ( $G$ ) of the system containing  $G_1$  which has the highest symmetry, and extend them to form a (closed or unclosed) polyhedron again. The fraction is easily verified to be always  $\frac{1}{2}$ ,  $\frac{1}{3}$ , or  $\frac{1}{6}$ .

Polyhedra whose symmetry-operations form a group such as  $G$  are called *holohedral* †; those whose symmetry-operations form a group such as  $G_1$  are called *merohedral* ‡ in general; or 'hemihedral,' 'tetartohedral,' or 'ogdohedral' § according as  $G_1$  contains  $\frac{1}{2}$ ,  $\frac{1}{3}$ , or  $\frac{1}{6}$  of the operations of  $G$ .

Similarly  $G$  is called a 'holohedral group,'  $G_1$  a 'merohedral' (or 'hemihedral,' &c.). The words 'holohedry,' 'merohedry,' &c., are often used for 'holohedral groups,' 'merohedral groups,' &c.

The groups  $T_d, C_{6v}, C_{4v}, C_{3v}, C_{2v}, C_2$  are often called '*hemimorphic* ¶.' Their polyhedra possess those faces of the corresponding holohedral polyhedra which meet *one* end of the 3-al axes, or 6-al, 4-al, 3-al, 2-al axis respectively.

The groups  $T_h, C_{6h}, C_{4h}, C_{3h}$  which have a symmetry-plane perpendicular to a symmetry-axis, but which are not holohedral, are sometimes called '*paramorphic*.'

The groups  $O, D_6, D_4, D_3, D_2$ , (or  $\bar{Q}$ ) are sometimes called '*enantiomorphous*.' They possess no operations of the second sort, and the corresponding polyhedra are not superposable on their own mirror-images in any plane. If  $p, p'$  are any two parallel planes equidistant from and on opposite sides of  $O$ ; and if  $F, F'$  are the polyhedra formed by planes equivalent to  $p, p'$  respectively in any one of these five classes; then  $F, F'$  are enantiomorphous to one another, and

\* This is true if the first subdivision of the hexagonal system is treated as a separate system, for  $C_3, C_{3i}, C_{3v}, D_3$  are subgroups of  $D_{3d}$ .

† From ὅλος, whole; ἔδρα, a face.

‡ μέρος, a part; ἔδρα, a face.

§ This only occurs in the case of  $C_3$ , and then only if the rhombohedral system is treated as a subdivision of the hexagonal.

¶ If the rhombohedral system is considered distinct from the hexagonal.

¶ Or 'Antimorphic.'

the faces of  $F$  and  $F'$  taken together make up the faces of the corresponding holohedral polyhedron.

§ 5. So far we have considered any face  $p$  of  $F$  to have a general position, and in that case each group has its own characteristic polyhedron (or 'form'), which has as many faces as the group has operations. If, however,  $p$  has a special position this is no longer the case. Thus, for example, if the plane  $p$  is perpendicular to a 4-al axis of  $O_h$  or  $O$ , or to a 2-al axis of  $T_d$ ,  $T_h$ , or  $T$ ; the form for all the groups  $O_h$ ,  $O$ ,  $T_d$ ,  $T_h$ , and  $T$  is a cube. If  $p$  is perpendicular to a 3-al axis of any of these groups, the form is a regular octahedron for  $O_h$ ,  $O$ , and  $T_h$ , and a regular tetrahedron for  $T_d$  and  $T$ .

These special forms are extremely common in the crystals found in nature, and the non-existence of general forms often makes it very difficult to determine the true symmetry of any given crystal.

## CHAPTER IX

THE CRYSTALLOGRAPHIC AXES AND  
AXIAL RATIOS.

§ 1. We shall in this chapter show how to choose most conveniently the position of the crystallographic axes of any crystal, and the magnitude of the axial ratios (§§ 2 to 8); and how to find the indices of all faces belonging to the same form as any face whose indices are given.

§ 2. For the triclinic system we choose lines parallel to *any* three edges \* as crystal axes, and *any* face as 'parametral face.' The axial ratios are perfectly general.

§ 3. In the monoclinic system the symmetry-axis (or the line perpendicular to the symmetry-plane in the case of  $C_2$ ) is always parallel to a possible edge, and in the plane perpendicular to it there is an indefinite number of lines parallel to possible edges (chap. iii, §§ 17 and 18).

Take then the symmetry-axis (the line perpendicular to the symmetry-plane) as the crystallographic axis  $OC$  † (cf. chap. ii, § 2, p. 9), and any two lines parallel to possible edges and perpendicular to  $OC$  (not in general at right angles to one another) as the axes  $OA$  and  $OB$ . Take *any* face as parametral face.

§ 4. In the classes  $D_2$ ,  $D_{2h}$  we take the three symmetry-axes as crystallographic axes, which is possible, for they are parallel to possible edges. For  $C_{2v}$  we take the symmetry-axis as the axis  $OC$ , and the intersections of the plane perpendicular to this axis (which is parallel to a possible face) with the symmetry-planes (which are also parallel to possible faces) as the other two crystallographic axes.

We take *any* face as parametral face.

\* It will be remembered that the crystallographic axes must of necessity be parallel to crystal edges (p. 9).

† In books on Descriptive Crystallography it is usual to take  $OB$  parallel to the symmetry-axis, instead of  $OC$  as in books on Theoretical Crystallography.

§ 5. For the tetragonal system we take the 4-al axis of the first or second sort as the crystallographic axis  $OC$ .

For  $C_4'$ ,  $C_4$  and  $C_{4h}$  we take a line parallel to any edge perpendicular to  $OC$  as the crystallographic axis  $OA$ ; for  $C_{4v}$  we take a line perpendicular to  $OC$  and lying in a symmetry-plane; and for  $D_{2d}$ ,  $D_4$ , and  $D_{4h}$  we take a 2-al rotation-axis. In each case we take the line with which  $OA$  would be made to coincide by a rotation through  $\frac{\pi}{2}$  about  $OC$  as the axis  $OB$ .

Now there is always a possible face making equal intercepts on  $OA$ ,  $OB$  (p. 42). Take such a face as parametral face; then the axial ratios may be written  $a:a:c$  ( $a=b$ ).

§ 6. The following method may be applied in the case of the rhombohedral system. Choose as the axis  $OC$ , for  $C_3$  and  $C_{3i}$  a line\* parallel to any edge, for  $C_{3v}$  and  $D_{3d}$  a line\* parallel to an edge and lying in a symmetry-plane, for  $D_3$  (and  $D_{3d}$ ) a line\* parallel to an edge and perpendicular to a 2-al axis. As the axes  $OA$ ,  $OB$  choose the positions into which  $OC$  would be brought by successive rotations through  $\frac{2\pi}{3}$  about the 3-al symmetry-axis.

Then assuming that there is a possible face perpendicular to the symmetry-axis†, we take this for parametral face, and may then write our axial ratios

$$a:a:a \ (a=b=c).$$

§ 7. For the hexagonal system as a whole, including the rhombohedral system as a subdivision, we take the 3-al or 6-al rotation-axis as the axis  $OC$ . For  $C_3$ ,  $C_{3i}$ ,  $C_{3h}$ ,  $C_6$ ,  $C_{6h}$  we take any line perpendicular to  $OC$  and parallel to a possible edge as the axis  $OA$ ‡; for  $C_{3v}$ ,  $C_{6v}$  we take a line perpendicular to  $OC$  and lying in a symmetry-plane; for  $D_3$ ,  $D_{3d}$ ,  $D_{3h}$ ,  $D_6$ ,  $D_{6h}$  we take a 2-al rotation-axis. We now take *two* more crystallographic axes  $OB$ ,  $OD$  (thus having four crystallographic axes in all) such that  $OA$  would be brought into the positions  $OB$ ,  $OD$  by successive rotations through  $\frac{2\pi}{3}$  about  $OC$ .

\* Not perpendicular to, or coinciding with the symmetry-axis.

† This can be *proved* for  $C_{3v}$ ,  $D_3$  and  $D_{3d}$  by the aid of §§ 17 and 18 of chap. iii.

‡ It is assumed (not proved) that there is such a line in the case of  $C_3$  and  $C_{3i}$ .

Let any face meet the axes  $OA, OB, OD$  in  $A', B', D'$  (Fig. 59).

$$\begin{aligned}\text{Then} \quad \frac{\sin(OB'D' - \frac{\pi}{3})}{\sin OB'D'} &= \frac{\sin OA'B'}{\sin OB'A'} = \frac{B'O}{OA'}, \\ \frac{\sin(OB'D' + \frac{\pi}{3})}{\sin OB'D'} &= \frac{\sin OD'B'}{\sin OB'D'} = \frac{B'O}{OD'}.\end{aligned}$$

Therefore by addition we have

$$\frac{1}{OA'} + \frac{1}{OB'} + \frac{1}{OD'} = 0.$$

Now there is a crystal face making equal intercepts on  $OA$  and  $OD$ ; therefore  $OA':OD'$  is always rational.

But

$$\frac{1}{OA'} + \frac{1}{OB'} + \frac{1}{OD'} = 0,$$

and hence  $OA':OB':OD'$  are rational ratios. Therefore the intercepts of any face on the four axes,  $OA,$

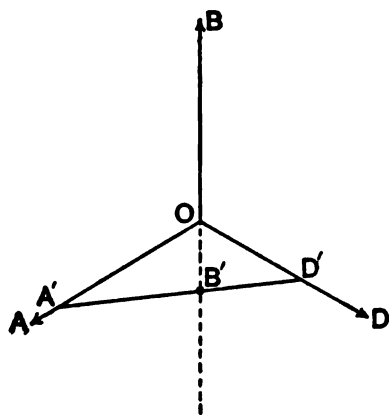


Fig. 59.

$OB, OD, OC,$  are proportional to  $\frac{a}{h}, \frac{a}{k}, \frac{a}{i}, \frac{c}{l}$  where  $a$  and  $c$  are some fixed quantities and  $h, k, i, l$  are integers. The ratios  $a:a:a:c$  are called the axial ratios, and  $h, k, i, l$  the indices of the face.

$$\text{Since} \quad \frac{1}{OA'} + \frac{1}{OB'} + \frac{1}{OD'} = 0, \quad h+k+i=0.$$

§ 8. In the regular system we take the three 4-al rotation-axes as crystallographic axes for  $O, O_h$ ; and for  $T, T_h, T_d$  we take the three 2-al rotation-axes. In either case there is a possible face perpendicular to each of the 3-al axes\*. Taking one of these as parametral face, we may write the axial ratios

$$a:a:a \quad (a=b=c).$$

§ 9. We thus see that we are able to choose as crystallographic axes lines similar to those which we have chosen in chap. vii as axes of coordinates.

\* This is assumed for  $T$  and  $T_h$ ; it may be proved for  $T_d, O,$  and  $O_h$ .

	Symbol of the crystal class *	Name of the class adopted in the rest of this book *	Number and name used by Professor H. A. Miers †	Number and name used by Professor E. S. Dana ‡§	Other names in general use (if any).	Number of faces in the general form.	Mineral typical of the class ‖.	Artificial crystal belonging to the class ‖.
Irregular	$C_1$ $C_1'$	Hemihedry Holoedry	i. Asymmetric ii. Centrosymmetric	32. Asymmetric 31. Normal	Hemipinakoidal, Pedial Pinakoidal	1 2	Axinite	$CaS_2O_3 \cdot 6H_2O$ $CaSO_4 \cdot 6H_2O$
	$C_2$	Hemihedry	iii. Equatorial	30. Clinohedral	Domestic	2	Clinohed-	$K_2S_4O_6$
	$T$	Tetartohedry	xxviii. Tesseral polar	6. Tetartohedral	Pentagon-dodecahedral	12	Ullmanite	$NaClO_3$
Regular	$T_h$	Paramorphic hemihedry	xxx. Tesseral central	2. Pyritohedral	Parallel-faced hemihedry, Pentagonal hemihedry, Dyakisdodecahedral	24	Pyrites	$KAl(SO_4)_3 \cdot 12H_2O$
	$T_d$	Hemimorphic hemihedry	xxxi. Ditetrasal polar	8. Tetrahedral	Hexakistetrahedral, Inclined hemihedry	24	Tetrahedrites Cuprite	$Na_2KAl(C_2O_4)_3 \cdot 4H_2O$ $NH_4Cl$
	$O$	Enantiomorphous hemihedry	xxix. Tesseral holoaxial	4. Plagihedral	Gyroidal hemihedry, Pentagon-icositetrahedral	24		
	$O_h$	Holoedry	xxxii. Ditetrasal central	1. Normal	Hexaktoctahedral	48	Galena, Fluor-spar	$As_2O_3$

\* Based on Schoenflies' notation.

† "Mineralogy," 1902.

‡ "A Textbook of Mineralogy," 1899.

§ For comparisons with other authors see Schoenflies' "Kristallssysteme und Kristallstruktur," p. 104, and "Zeitschr. f. Kryst. u. Min.," xxiv, p. 280.

|| From Groth's "Physikalische Kristallographie."



Let any crystal face whose indices are  $h, k, l$  meet these axes in  $A', B', C'$ , and the perpendicular from  $O$  on the face in  $M'$  (Fig. 10, p. 17); and let the Fedorow coordinates of the point  $M'$  be  $\xi' \eta' \zeta'$ .

$$\text{Then} \quad h:k:l = \frac{a}{OA'} : \frac{b}{OB'} : \frac{c}{OC'}$$

$$= a \cos M'OA' : b \cos M'OB' : c \cos M'OC' = a\xi' : b\eta' : c\zeta'.$$

Now an operation which brings the face  $(hkl)$  to coincide with some other face  $p$  brings  $M'$  to coincide with the perpendicular from  $O$  on  $p$ . Hence to find the indices of all faces equivalent to (belonging to the same form as) the face  $(hkl)$ , it is sufficient to know the Fedorow coordinates of all points equivalent to the point  $(\xi' \eta' \zeta')$ . These are given in chapter vii. Using the systems of coordinates there given and the axial ratios obtained in this chapter, we find at once that in any class *the system of indices of faces equivalent to  $(hkl)$  is of exactly the same type as the system of Fedorow coordinates of points equivalent to  $(\xi \eta \zeta)$ .*

Thus as an example take the class  $C_4$ . The Fedorow coordinates of a system of equivalent points are (p. 85)

$$\xi \eta \zeta, -\eta \xi \zeta, -\xi -\eta \zeta, \eta -\xi \zeta.$$

Therefore the system of faces equivalent to  $(hkl)$  is, since  $b = a$ ,  $(a\xi' a\eta' c\zeta')$ ,  $(a-\eta' a\xi' c\zeta')$ ,  $(a-\xi' a-\eta' c\zeta')$ ,  $(a\eta' a-\xi' c\zeta')$  or  $(hkl)$ ,  $(\bar{k}hl)$ ,  $(\bar{h}\bar{k}l)$ ,  $(k\bar{h}l)$ .

The symbol  $\{hkl\}$  is often used to denote the form to which the face  $(hkl)$  belongs.

Again, we have another proof of the fact that in the hexagonal system  $h+k+i = 0$  (p. 96); for

$$\xi'_1 + \xi'_2 + \xi'_3 = 0 \text{ and } h:k:i = \xi'_1 : \xi'_2 : \xi'_3.$$

§ 10. We sum up the crystal classes in the appended table. The first three columns contain the names and symbols adopted in the rest of this book; they are based on Schoenflies' notation.

The symmetry of some of the crystals in the last two columns of the table is not quite certain.

## CHAPTER X

THE DEPENDENCE OF PHYSICAL PROPERTIES OF  
CRYSTALS ON SYMMETRY.

§ 1. Physical phenomena of solids are of three kinds, scalar, vectorial, and tensorial. Scalar phenomena are those which can be expressed in terms of magnitude, but not of direction; well-known examples are temperature, density, electrical potential, and sublimation vapour pressure.

Vectorial phenomena are those which can be expressed in terms of magnitude and direction, and can therefore be geometrically represented by a finite straight line of known length and direction; examples are transmission of light, thermal and electrical conductivity, &c.

Tensorial phenomena are those which can be expressed in terms of magnitude and two opposed directions, and can be geometrically represented by two finite straight lines equal, and in opposite directions; such phenomena are strain and stress.

§ 2. The usual problems we have to consider in crystal-physics consist in determining the dependence of one phenomenon ( $A$ ) on another ( $B$ ) when the material in which the phenomena are manifested is crystalline; thus e.g. it may be required to find the electrical current which passes along a thin rod (of known length and cross section) which is cut from a crystal *in any given direction*, when the difference of potential at the ends of the rod is given.

If both  $A$  and  $B$  are scalar (e.g. if we are considering the influence of temperature on density) we can draw no distinction between amorphous and crystalline media.

§ 3. If  $B$  is scalar, and  $A$  vectorial (e.g. if we are considering the property of pyroelectricity, i.e. the appearance of electrical

polarity on crystals with change of temperature), a distinction between amorphous and crystalline matter at once arises.

Let us take any three orthogonal reference-axes  $OX$ ,  $OY$ ,  $OZ$ . Now  $A$  can be represented by a straight line through  $O$ ; let its length be  $\rho$  and its direction-cosines  $l$ ,  $m$ ,  $n$ ; then  $A$  may be resolved into three components of magnitudes,  $\rho l$ ,  $\rho m$ ,  $\rho n$ , along  $OX$ ,  $OY$ , and  $OZ$ :  $B$ , being scalar, is represented by a magnitude  $\theta$ .

Then we have relations of the form \*

$$\left. \begin{aligned} \rho l &= l_1 \theta \\ \rho m &= l_2 \theta \\ \rho n &= l_3 \theta \end{aligned} \right\},$$

where  $l_1$ ,  $l_2$ ,  $l_3$  are constants depending on the nature of the medium and the positions of the reference-axes.

Consider now the ellipsoid  $\frac{x^2}{l_1^2} + \frac{y^2}{l_2^2} + \frac{z^2}{l_3^2} = 3\theta^2$ ; the length of that radius vector whose direction cosines are  $l$ ,  $m$ ,  $n$  is

$$\sqrt{3\theta^2 \div \left( \frac{l^2}{l_1^2} + \frac{m^2}{l_2^2} + \frac{n^2}{l_3^2} \right)} = \rho.$$

Hence the relation between the direction and magnitude of the straight line, which represents  $A$  when  $\theta$  is given, is geometrically represented by the relation between the direction and magnitude of the radius vector from the centre of a certain ellipsoid. The dimensions and orientation of this ellipsoid must be independent of our reference-axes, for it represents a physical property of the crystalline material.

§ 4. If  $A$  and  $B$  are both vectorial (e.g. if we are considering the dependence of electrical current on electromotive force), let the lengths and direction-cosines of the straight lines representing  $A$  and  $B$  be respectively  $\rho$ ,  $l$ ,  $m$ ,  $n$ ;  $R$ ,  $L$ ,  $M$ ,  $N$ . The components of these lines along the reference-axes are  $\rho l$ ,  $\rho m$ ,  $\rho n$ ;  $RL$ ,  $RM$ ,  $RN$ . Now in a very large number of important cases it is found that  $\rho l$ ,  $\rho m$ ,  $\rho n$  can be expressed linearly in terms of  $RL$ ,  $RM$ ,  $RN$ ; at least to a first approximation.

We have then equations of the form

$$\left. \begin{aligned} \rho l &= a_{11}RL + a_{12}RM + a_{13}RN \\ \rho m &= a_{21}RL + a_{22}RM + a_{23}RN \\ \rho n &= a_{31}RL + a_{32}RM + a_{33}RN \end{aligned} \right\} \quad . \quad . \quad . \quad (i).$$

\* These linear relations are in most cases found to hold good, at any rate as a first approximation.

## 100 THE DEPENDENCE OF PHYSICAL PROPERTIES

Now again in some cases we can prove by thermodynamical considerations \*, or failing these experimentally, that

$$a_{12} = a_{21}, a_{31} = a_{13}, a_{23} = a_{32}.$$

If that is so, consider the ellipsoid

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{23}yz + 2a_{31}zx + 2a_{12}xy = 1 \quad \text{(ii)}.$$

Let  $r$  be the length of that radius vector whose direction-cosines are  $L, M, N$ ; then the tangent plane at the extremity of this radius vector is

$$x(a_{11}L + a_{12}M + a_{31}N) + y(a_{12}L + a_{22}M + a_{23}N) + z(a_{31}L + a_{23}M + a_{33}N) = \frac{1}{r},$$

$$\text{or} \quad xl + ym + nz = \frac{R}{\rho} \cdot \frac{1}{r}.$$

The perpendicular ( $p$ ) from the centre on this plane has the direction-cosines  $l, m, n$  and is of length  $\frac{R}{\rho} \cdot \frac{1}{r}$ ; so that the ellipsoid (ii) gives us the following method of finding geometrically the relation between the lines representing  $A$  and  $B$ . Take the radius vector ( $r$ ) whose direction-cosines are those of the line (of length  $R$ ) representing  $B$ , and draw a tangent plane at its extremity; then the straight line which represents  $A$  is parallel to the perpendicular ( $p$ ) from the centre on this tangent plane, and its length  $\rho$  is given by the relation  $\rho pr = R$ .

It should be noticed that if  $B$  is of such a magnitude that it is represented (in magnitude and direction) by a radius vector of the ellipsoid (ii),  $A$  is represented by the corresponding vector of the polar reciprocal of (ii) with respect to a concentric sphere of unit radius.

As before the dimensions and orientation of the ellipsoid (ii) cannot depend on the position of the axes of reference.

Even if we do not know that in equations (i)

$$a_{12} = a_{21}, a_{31} = a_{13}, a_{23} = a_{32}$$

we can represent the physical property under discussion *partially* by means of an ellipsoid. For squaring and adding equations (i) we have

$$\rho^2 = R^2 (aL^2 + bM^2 + cN^2 + 2fMN + 2gNL + 2hLM),$$

\* e.g. when we discuss the dependence of the electrical moment of a dielectric on the electrostatic field.

where  $a = a_{11}^2 + a_{21}^2 + a_{31}^2$ ,  $f = a_{12} \cdot a_{13} + a_{22} \cdot a_{23} + a_{32} \cdot a_{33}$ , &c., since  $l^2 + m^2 + n^2 = 1$ .

Hence  $\frac{R}{\rho} =$  the length of the radius vector of the ellipsoid

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + hxy = 1 \quad \dots (iii),$$

drawn from the centre parallel to the straight line representing  $B$ .

Since this ellipsoid only represents the connexion between the magnitude and direction of  $B$  with the *magnitude* (not the direction) of  $A$ , the geometrical representation of the property under discussion is not so complete as in the case when  $a_{12} = a_{21}$ ,  $a_{31} = a_{13}$ ,  $a_{23} = a_{32}$ .

The dimensions and orientation of the ellipsoid (iii) do not depend on the choice of axes.

§ 5. We mentioned before\* that all physical properties of a given crystal must share its symmetry, and that a crystal cannot have a higher symmetry than the least symmetrical of its physical properties. In general the physical properties are of a higher order of symmetry than the geometrical (the occurrence of 'forms'), but the true symmetry of the latter is not always easy to determine, partly owing to the frequent non-occurrence of any but special forms on crystals found in nature, and partly owing to the simulation by crystals of higher symmetry than they really possess by means of twinning, &c. (see next chapter).

In those cases where the mutual dependence of two phenomena can be represented by an ellipsoid, the ellipsoid must possess the symmetry of the crystal. Now the elements of symmetry possessed by *every* ellipsoid are three mutually perpendicular 2-al rotation-axes, a centre of symmetry, and three planes of symmetry perpendicular to the axes (in fact the symmetry-elements of the group  $D_{2h}$ ). An ellipsoid cannot possess a 3-al, 4-al, &c., axis unless it is a figure of revolution about that axis; it cannot possess more than one axis other than 2-al axes unless it is a sphere.

§ 6. We can therefore divide crystals into five classes by means of their behaviour towards those physical properties which can be represented by an ellipsoid. Such a property is the propagation of light (for it can be represented by the Fresnel ellipsoid), and we shall use that as our illustration.

(1) In the regular system the ellipsoid must be a sphere;

\* p. 36.

all crystals of the regular system are 'singly refracting,' or 'isotropic as regards light \*.'

(2) In the hexagonal, tetragonal, and rhombohedral systems the ellipsoid must be a figure of revolution whose axis coincides with the 6-al, 4-al, or 3-al symmetry-axis; all crystals of these systems are uniaxial †.

(3) In the orthorhombic system the axes of the ellipsoid must coincide with those directions which we took as axes of reference in chapters vii and ix (pp. 84 and 94); all crystals of the orthorhombic systems are biaxial †, and the directions of the principal axes of the Fresnel ellipsoid are the same for every wave length of light.

(4) In the monoclinic system *one* of the axes of the ellipsoid must coincide with the axis of symmetry or be perpendicular to the symmetry-plane; all crystals of the monoclinic system are biaxial, and the direction of *one* of the principal axes of the Fresnel ellipsoid is the same for every wave length of light, but the directions of the other two axes in general vary with the wave length.

(5) In the triclinic system there is no position with which an axis of the ellipsoid must necessarily coincide; all crystals of this class are biaxial, and the directions of all the axes of the Fresnel ellipsoid vary with the wave length.

§ 7. Consider now a more general case. Suppose that the phenomena  $A, B$  are represented by straight lines  $Oa, Ob$  whose lengths are  $\rho, R$ , and direction-cosines (referred to orthogonal axes  $OX, OY, OZ$ )  $l, m, n$ ;  $L, M, N$ ; and suppose that we have relations of the form

$$\left. \begin{aligned} \rho l &= f_1(RL, RM, RN) \\ \rho m &= f_2(RL, RM, RN) \\ \rho n &= f_3(RL, RM, RN) \end{aligned} \right\} \dots \dots \dots (i),$$

whereby  $\rho l, \rho m, \rho n$  are expressed algebraically in terms of  $RL, RM, RN$  and certain constants (corresponding to the  $a'$  of equations (i) in § 4) which depend only on the nature of the crystalline medium and the choice of reference-axes.

Now in a medium of given symmetry we can deduce certain relations between these constants as follows.

Let any symmetry-operation of the medium bring  $Oa, Ob, OX, OY, OZ$  into the positions  $Oa', Ob', OX', OY', OZ'$  respec-

\* Sometimes called simply 'isotropic,' but it seems better to reserve this term for those solids whose behaviour towards *all* physical phenomena can be represented by a sphere; i.e. glassy or amorphous bodies.

† Or uniaxal, biaxal; both methods of spelling are in use.

tively; and let the direction-cosines of  $Oa'$ ,  $Ob'$  referred to the axes  $OX$ ,  $OY$ ,  $OZ$  be  $l'$ ,  $m'$ ,  $n'$ ;  $L'$ ,  $M'$ ,  $N'$ .

Then since, if  $B$  is represented by  $Ob$ ,  $A$  is represented by  $Oa$ ; therefore by the symmetry of the crystal if  $B$  is represented by  $Ob'$ ,  $A$  must be represented by  $Oa'$ ; and hence we must have

$$\left. \begin{aligned} \rho l' &= f_1(RL', RM', RN') \\ \rho m' &= f_2(RL', RM', RN') \\ \rho n' &= f_3(RL', RM', RN') \end{aligned} \right\} \dots \dots \dots (ii).$$

Now by the formulae of chap. vii, §§ 8 and 9 (pp. 82, 83), we can express the direction-cosines of  $Oa'$ ,  $Ob'$  referred to  $OX$ ,  $OY$ ,  $OZ$  (i.e.  $l'$ ,  $m'$ ,  $n'$ ;  $L'$ ,  $M'$ ,  $N'$ ) in terms of the direction-cosines of  $Oa$ ,  $Ob$  referred to  $OX$ ,  $OY$ ,  $OZ$  (i.e.  $l$ ,  $m$ ,  $n$ ;  $L$ ,  $M$ ,  $N$ ); we have in fact relations of the form

$$\left. \begin{aligned} l &= C_{11}l' + C_{12}m' + C_{13}n' \\ m &= C_{21}l' + C_{22}m' + C_{23}n' \\ n &= C_{31}l' + C_{32}m' + C_{33}n' \\ L &= C_{11}L' + C_{12}M' + C_{13}N' \\ M &= C_{21}L' + C_{22}M' + C_{23}N' \\ N &= C_{31}L' + C_{32}M' + C_{33}N' \end{aligned} \right\} \dots \dots \dots (iii),$$

where every  $C$  is known.

Substituting in (i) from (iii) we have

$$\begin{aligned} C_{11}\rho l' + C_{12}\rho m' + C_{13}\rho n' &= f_1(C_{11}RL' + C_{12}RM' + C_{13}RN', \\ C_{21}RL' + C_{22}RM' + C_{23}RN', C_{31}RL' + C_{32}RM' + C_{33}RN') \\ &= \phi_1(RL', RM', RN') \text{ say} \end{aligned}$$

and two similar equations.

$$\text{Solving these we have (remembering that } \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{vmatrix} = \pm 1)$$

$$\rho l' = \pm \begin{vmatrix} \phi_1(RL', RM', RN') & C_{12} & C_{13} \\ \phi_2(RL', RM', RN') & C_{22} & C_{23} \\ \phi_3(RL', RM', RN') & C_{32} & C_{33} \end{vmatrix}$$

and two similar equations  $\dots \dots \dots (iv).$

These equations (iv) must be identical with (ii); and therefore

$$f_1(RL', RM', RN') \equiv \pm \begin{vmatrix} \phi_1(RL', RM', RN') & C_{12} & C_{13} \\ \phi_2(RL', RM', RN') & C_{22} & C_{23} \\ \phi_3(RL', RM', RN') & C_{32} & C_{33} \end{vmatrix}$$

and two similar identities.

Expanding  $f_1, f_2, f_3, \phi_1, \phi_2, \phi_3$  in powers and products of

powers of  $RL'$ ,  $RM'$ ,  $RN'$ , and equating coefficients on the two sides of these three equations, we get some of the relations between the constants which we required; we obtain the remainder by using other symmetry-operations of the medium. Not all the relations so obtained are necessarily independent.

§ 8. When either  $A$  or  $B$  is a tensorial phenomenon the mathematical relations are more complicated, since in this case the phenomenon cannot well be represented by the resultant of three components. For these and other developments of this branch of the subject we must refer the reader to W. Voigt's "*Elemente der Krystallophysik*," Th. Liebisch's "*Physikalische Krystallographie*," &c. For instance, it is there shown that the elasticity of a triclinic crystal can be represented by a surface involving 21 constants. A property of considerable importance in theories of crystal-structure, and one which may have some connexion with elasticity, is that of cleavage, or the tendency of crystals to break in such a way that the surface of fracture is a plane. This plane is always a possible face of the crystal, and the crystal may with equal ease be broken so that any other face of the same form is the fracture-surface; in other words the cleavage shares the symmetry of the crystal. Very often two (or more) faces belonging to different forms can be obtained by fracture in this way, but not in general with equal ease. Forms obtained by cleavage may or may not occur as forms found on a crystal grown under the ordinary conditions. The mathematical problems connected with cleavage (e.g. its connexion with elasticity) still await solution.

## CHAPTER XI

## ON THE GROWTH OF CRYSTALS.

§ 1. In this chapter we propose to give a very brief account of the phenomena observed in processes connected with the growth of crystals. As is well known, a material crystallizes out from a supersaturated solution, from a liquid cooled to its freezing-point, or from a supersaturated vapour\*.

We shall first consider a crystal which has grown without any extraneous disturbance, and shall assume that the polyhedron formed by all faces of such a crystal has the full symmetry of the ideal crystal of p. 36; in other words we shall assume that all faces whose physical properties are the same are equidistant from a fixed point which we have called the centre of the crystal. The faces of the crystal do not all necessarily belong to the same 'form' (i. e. are not all equivalent to one another), but if one face of a form appears all the rest must appear also and be equidistant from the centre. Faces belonging to the same form must be alike in all their physical properties; but faces belonging to different forms differ in hardness, lustre, &c., and are not in general at the same distance from the centre. A crystal may possess faces belonging only to one form, if that form is closed; otherwise the faces must belong to two or more forms†. Even if the symmetry allows of all the faces belonging to a single closed form, very often the crystal possesses faces belonging to several different forms.

§ 2. Among other properties which are different for faces belonging to different forms must be the tension (due to capillary action) of the surfaces separating a crystal from the liquid or vapour in which it is growing. This tension may

\* This process is called 'sublimation'; good examples are camphor, phosphorus, and ammonium chloride.

† For example, crystals whose symmetry is that of  $C_4$  must possess faces belonging to 4 forms, crystals whose symmetry is that of  $C_2$  must possess faces belonging to 2 forms, crystals whose symmetry is that of  $C_3$  must possess faces belonging to 3 forms, &c.

be measured by a constant for each face which we shall call the capillarity constant. A crystal must always grow (if undisturbed) so that its surface energy is a minimum for a given volume of solid matter crystallized out; hence if  $s_1, s_2, s_3, \dots$  are the areas of the various faces,  $k_1, k_2, k_3, \dots$  the corresponding capillarity constants, then  $E \equiv k_1 s_1 + k_2 s_2 + k_3 s_3 + \dots$  must be a minimum for a given volume of the crystal\*. There is an interesting connexion between the capillarity constants of the various faces and their distance from the centre of the crystal†, which may be expressed in the following theorem:—

*The perpendiculars on the faces of a crystal from a certain point within it are proportional to the capillarity constants of the faces.*

Let  $I$  be any point within the crystal and let  $n_1, n_2, n_3, n_4 \dots$  be the perpendiculars from  $I$  on the faces  $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \dots$  whose areas are  $s_1, s_2, s_3, s_4, \dots$ , and whose capillarity constants are  $k_1, k_2, k_3, k_4, \dots$ . Let  $V$  be the volume of the crystal, and let  $P_1, P_2, P_3, P_4, \dots$  be the planes through  $I$  parallel to  $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \dots$ . Now there are certain geometrical relations between the quantities  $s_1, s_2, s_3, s_4, \dots$  which depend on the fact that they are the areas of polyhedron faces whose normals have fixed directions.

For let  $A$  be any point;  $n_{1a}, n_{2a}, n_{3a}, n_{4a}, \dots$  the perpendiculars from  $A$  on  $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \dots$ ;  $a_1, a_2, a_3, a_4 \dots$  ‡ the perpendiculars from  $A$  on  $P_1, P_2, P_3, P_4, \dots$ .

Then  $n_1 - n_{1a} = a_1, n_2 - n_{2a} = a_2, \&c.$

Since  $V = \frac{1}{3} \sum n s = \frac{1}{3} \sum n_a s,$

$$\therefore \sum (n - n_a) s = 0,$$

and  $\therefore \sum a s = 0.$

$a_1, a_2, a_3, a_4, \dots$  are not independent, for the position of  $A$  is completely defined by the perpendiculars from it to the three planes  $P_1, P_2, P_3$ . We must have linear relations of the form  $a_4 = p_4 a_1 + q_4 a_2 + r_4 a_3, a_5 = p_5 a_1 + q_5 a_2 + r_5 a_3, \&c.$

Now take other points  $B, C, D, \dots$  such as  $A$ ; then we shall have an indefinite number of similar relations

$$\sum b s = \sum c s = \sum d s = \dots = 0.$$

\* Of course if two or more of the faces belong to the same form the corresponding areas and capillarity constants must be equal.

† If the growth of the crystal is undisturbed.

‡  $a_1$  is reckoned positive or negative according as  $A$  is on the same or on the opposite side of  $P_1$  as  $\sigma_1$ , and so for  $a_2, a_3, a_4, \dots$

These are, however, only equivalent to *three* independent relations.

For choose quantities  $\lambda, \mu, \nu$  to satisfy the equations

$$d_1 = \lambda a_1 + \mu b_1 + \nu c_1; \quad d_2 = \lambda a_2 + \mu b_2 + \nu c_2; \quad d_3 = \lambda a_3 + \mu b_3 + \nu c_3.$$

$$\begin{aligned} \text{Then } d_4 &= p_4 d_1 + q_4 d_2 + r_4 d_3 = p_4(\lambda a_1 + \mu b_1 + \nu c_1) \\ &\quad + q_4(\lambda a_2 + \mu b_2 + \nu c_2) + r_4(\lambda a_3 + \mu b_3 + \nu c_3) \\ &= \lambda(p_4 a_1 + q_4 a_2 + r_4 a_3) + \mu(p_4 b_1 + q_4 b_2 + r_4 b_3) \\ &\quad + \nu(p_4 c_1 + q_4 c_2 + r_4 c_3) = \lambda a_4 + \mu b_4 + \nu c_4. \end{aligned}$$

Similarly

$$d_5 = \lambda a_5 + \mu b_5 + \nu c_5, \text{ \&c., and } \Sigma ds \equiv \lambda \Sigma as + \mu \Sigma bs + \nu \Sigma cs.$$

Suppose now the crystal to receive a small deformation, such that the normals to its faces are fixed in direction; then the volume of matter added on a face  $\sigma$  lies between  $sdn$  and  $(s+ds)dn$  and therefore  $= sdn$ , neglecting small quantities of the second order.

Hence  $dV = \Sigma sdn$ .

$$\begin{aligned} \text{Again } dV &= d\left\{\frac{1}{3}\Sigma ns\right\} = \frac{1}{3}\Sigma\{nds + sdn\} = \frac{1}{3}[\Sigma\{nds\} + dV] \\ \therefore \Sigma nds &= 2dV. \end{aligned}$$

Now if the surface energy  $E \equiv \Sigma ks$  is a minimum when  $V$  is constant, we must have  $\Sigma nds = \Sigma kds = 0$ ; and also we have  $\Sigma ads = \Sigma bds = \Sigma cds = 0$ .

Since the quantities  $ds_1, ds_2, ds_3, ds_4, \dots$  are connected by these five equations, and by these only, we have by the ordinary rules of the differential calculus

$$\begin{aligned} n_1 - aa_1 - \beta b_1 - \gamma c_1 &= \rho k_1 \\ n_2 - aa_2 - \beta b_2 - \gamma c_2 &= \rho k_2 \\ n_3 - aa_3 - \beta b_3 - \gamma c_3 &= \rho k_3 \\ &\dots \dots \dots \end{aligned}$$

where  $a, \beta, \gamma, \rho$  are (unknown) constants.

Now choose a point  $I'$  whose distances from  $P_1, P_2, P_3$  are respectively  $aa_1 + \beta b_1 + \gamma c_1, aa_2 + \beta b_2 + \gamma c_2, aa_3 + \beta b_3 + \gamma c_3$ ; then its distance from  $P_4$  is

$$\begin{aligned} p_4(aa_1 + \beta b_1 + \gamma c_1) + q_4(aa_2 + \beta b_2 + \gamma c_2) + r_4(aa_3 + \beta b_3 + \gamma c_3) \\ = a(p_4 a_1 + q_4 a_2 + r_4 a_3) + \beta(p_4 b_1 + q_4 b_2 + r_4 b_3) \\ + \gamma(p_4 c_1 + q_4 c_2 + r_4 c_3) = aa_4 + \beta b_4 + \gamma c_4. \end{aligned}$$

Similarly its distance from  $P_5$  is  $aa_5 + \beta b_5 + \gamma c_5$ , &c.

Let its distances from  $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \dots$  be  $n'_1, n'_2, n'_3, n'_4 \dots$ , then

$$\begin{aligned} n'_1 &= n_1 - aa_1 - \beta b_1 - \gamma c_1, \quad n'_2 = n_2 - aa_2 - \beta b_2 - \gamma c_2, \\ n'_3 &= n_3 - aa_3 - \beta b_3 - \gamma c_3, \quad n'_4 = n_4 - aa_4 - \beta b_4 - \gamma c_4, \text{ \&c.} \\ \therefore n'_1 : n'_2 : n'_3 : n'_4 : \dots &= k_1 : k_2 : k_3 : k_4 : \dots^* \end{aligned}$$

\* In some cases it is easier (instead of using this geometrical property to

Since  $k$  is the same for all faces belonging to the same form,  $I'$  must be equidistant from all faces of the same form.

It may prove impossible to find a point whose distances from the faces  $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \dots$ , are in the ratios  $k_1 : k_2 : k_3 : k_4 : \dots$ ; or even if it is possible, it may be found that  $E$  is a maximum, not a minimum. That means that a crystal of the material considered cannot grow with this particular combination of faces, but must grow with some other combination.

The capillarity constants vary with the medium with which the crystal is in contact; hence it is easy to account for the observed fact that the relative predominance of different faces on crystals grown from a solution often alters if a foreign substance is added to the solution.

§ 3. Suppose that when the crystal is very small (when it is just beginning to grow) its position is  $J$ ; and let the thicknesses of the crusts of the solid (measured parallel to the normals) deposited on the faces  $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \dots$  in a small period of time be in the ratios  $h_1 : h_2 : h_3 : h_4 : \dots$ .

Now, assuming that the rates of growth parallel to the normals of the different faces are always in constant ratios\*, it is evident that the faces of the crystal in *all* stages of its growth are at distances from  $J$  which are in the ratios of  $h_1 : h_2 : h_3 : h_4 : \dots$ .

From considerations of symmetry the rates of growth on all faces belonging to the same form must be equal, and therefore  $J$  must be always equidistant from faces of the same form.

In the case of crystals belonging to twenty-two of the thirty-two classes there is only one point which is always equidistant from all faces of the same form; the points  $J$  and  $I'$  (§ 2) must coincide at the 'centre' of the crystal, and we must have  $h_1 : h_2 : h_3 : h_4 : \dots = k_1 : k_2 : k_3 : k_4 : \dots$ . In the case of crystals belonging to the classes  $C_1, C_2, C_3, C_{2v}, C_3, C_{3v}, C_4, C_{4v}, C_6, C_{6v}$  there are an indefinite number of points equidistant from all faces of the same form†; in these classes

determine the shape of the crystal, assuming the capillarity constants known) to write down  $E$  and  $V$  in terms of suitably chosen independent variables; and to find what values of these variables make  $E$  a minimum and  $V$  constant, by the ordinary methods of the differential calculus.

\* Just as we assumed that the capillarity constants depend only on the directions and not on the sizes of the faces.

† In the case of  $C_1$  any point, in the case of  $C_2$  any point on the symmetry-plane, and in the case of the other eight classes any point on the symmetry-axis is equidistant from all faces of the same form. The position of the

$J$  does not necessarily coincide with  $I'$ , and we have no proof that  $h_1:h_2:h_3:h_4:\dots = k_1:k_2:k_3:k_4:\dots$

G. Wulff\* has attempted to compare the capillarity constants of different faces belonging to monoclinic crystals of ferrous ammonium sulphate by measuring the rates of growth on these faces. Elaborate precautions were necessary in order to eliminate the effects of disturbing forces during growth.

§ 4. Such crystals as we have been considering are those ideal crystals which may be supposed grown indefinitely slowly at constant temperature. Crystals found in nature have very rarely been grown under circumstances which even approximate to those necessary for ideal growth.

Take for instance a crystal grown from solution. The latter is usually considerably supersaturated, there is consequently a rapid deposition of solid on the faces of the crystal; the portion of liquid nearest the crystal becomes less supersaturated and consequently less dense than the portions further away; and before diffusion has time to make good the difference, convection currents are set up in the liquid, which exercise a disturbing influence on the growth of the crystal. Moreover, very often a crystal in growing comes in contact with other solid bodies and has not free room to develop.

§ 5. We have based our conclusions in previous chapters on the 'law of rational indices.'

Sometimes however, at least during the process of growth, faces make their appearance which are inclined at a small but measurable angle (a few minutes) to the position which would give them simple rational indices. Thus Professor Miers has shown that growing crystals of potash alum belonging to the regular system, which at first glance appear to have only the eight faces belonging to the form  $\{111\}$ †, have in reality twenty-four faces. These are arranged in threes making small angles with the positions which would be occupied by the faces of  $\{111\}$ ; and they make triangular pyramids of very small height, of which the faces of  $\{111\}$  would be the bases.

Such faces are called *Vicinal faces*; and their arrangement accords with the general symmetry of the crystal as shown

'centre' of a crystal belonging to one of the above-mentioned ten classes is to some extent arbitrary.

\* "Zeitschr. f. Kryst. u. Min.," xxxiv, 474.

† i. e. which seem to have the shape of a regular octahedron.

by Professor Miers in the case of alum\*, and by C. Viola in the case of calcite†. This fact renders untenable the hypothesis of G. Wulff‡ that they are due to a compromise between the molecular forces of the crystal trying to arrange particles deposited on it in faces having rational indices, and the forces due to the convection currents mentioned in § 4; for of course these convection currents do not partake of the symmetry of the crystal§.

§ 6. The arrangement of the vicinal faces may be a valuable help in determining the symmetry of a crystal in those cases where the presence of none but special forms makes it difficult to determine the symmetry otherwise. Since, however, they are not of universal occurrence, or at all easy to recognize, use is far more frequently made of the closely allied *etched-figures*.

When we expose a crystal face to the action of any solvent,

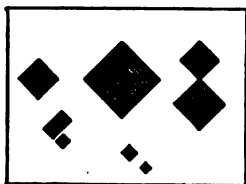


Fig. 60. Galena (class  $O_h$ ) etched on a face perpendicular to a 4-al axis with cold HCl. (After F. Becke¶).

it is found that solution does not take place uniformly over the whole surface, but that shallow pits are formed all over the faces, whose sides are formed of planes which are vicinal faces (see Fig. 60). These pits can be readily seen under the microscope||; they have the symmetry of the crystal; and it is only by means of these etched-figures that the symmetry of certain crystals has been determined.

§ 7. Crystals growing out of the same vapour or liquid do not grow independently, but exercise a mutual influence on one another.

If the deposition of solid matter takes place only slowly there is often a tendency on the part of the larger crystals to

\* "Report of the British Association," 1894, p. 654.

† "Zeitschr. f. Kryst. u. Min.," xxxv, 282.

‡ "Zeitschr. f. Kryst. u. Min.," xxxiv, 461.

§ For though their direction and magnitude are determined partly by the molecular forces of the crystal and by the angles between its faces which partake of the symmetry of the crystal, they are also determined by the action of gravity which does not partake of this symmetry. *See also, Proc. Roy. Soc.*

|| The solvent should usually be applied for a very short time, and acids used as solvents should be fairly dilute; however, it is difficult to lay down general rules; experiments must be made to determine the best treatment for each kind of crystal.

¶ See Th. Liebisch's "Grundriss der physikalischen Krystallographie," p. 44.

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grow at the expense of the smaller (which are redissolved); this is a result of the tendency towards a minimum of surface energy, as explained in § 2; for a given volume of solid has a smaller surface energy when it is collected into one big crystal, than when it is divided up into smaller ones\*.

§ 8. Even, however, in those cases where the growth is too rapid, or some other reason prevents this 'eating up' of the smaller crystals by the larger, crystals may influence the growth of their neighbours. Thus we often have several crystals side by side, united along planes or partially interpenetrating, such that the corresponding faces of all the crystals are parallel. This arrangement is called *parallel grouping* (see Fig. 61); it is found sometimes even in the case of two crystals of *different* material but similar form.

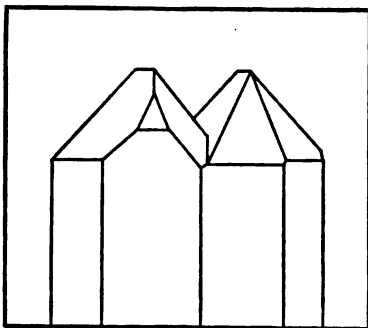


Fig. 61. Two quartz crystals in parallel position.

§ 9. Still more interesting is the case of two crystals which are not parallel, but which grow together in such a way that if one of them were to be rotated through an angle  $\pi$  about a certain line  $l$ , or reflected in a certain plane  $p$ , it would be parallel to the other. Such crystals are said to be 'twinned.'

The line  $l$  may be (1) normal to a possible face of both crystals; (2) parallel to a possible edge; (3) parallel to a possible face and perpendicular to a possible edge (the occurrence of this last case is, however, doubtful). The plane  $p$  must be parallel to a possible face.

*Twinned crystals* or *twins* may be united along a plane (called the 'composition plane'), and are then called 'contact.'

\* Suppose we have two crystals of the same shape and of lengths  $a_1, a_2$ ; and suppose that one 'eats up' the other, making one crystal of the same shape and of length  $b$ . Then  $a_1^3 + a_2^3 = b^3$ . Now the ratio of the surface energy of the single crystal to the sum of the surface energies of the two component crystals viz.  $b^2 : a_1^2 + a_2^2$  is  $< 1$ ; for

$$b^2 = (a_1^3 + a_2^3)^{2/3} < a_1^2 + a_2^2 + 6a_1a_2. \quad a_1a_2 < a_1^2 + a_2^2 + 3a_1a_2(a_1^2 + a_2^2) < (a_1^3 + a_2^3)^{2/3}.$$

Hence so long as two crystals of the same sort exist together, the surface energy can be diminished by letting one eat up the other.

or 'juxtaposition twins' (Fig. 62); or they may partially or wholly interpenetrate, the surface separating them being of irregular shape, and are then called 'penetration twins' (Fig. 63). Very often we have a line or ring of crystals, the second twinned on the first and third, the third on the second

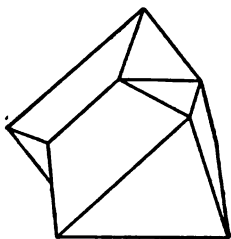


Fig. 62. Spinel twin\* (class  $O_h$ ; line  $l$  perpendicular to  $(11\bar{1})$ ).

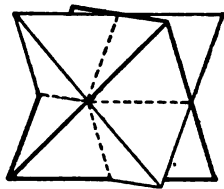


Fig. 63. Twin of two tetrahedra\* (class  $T_d$ ; line  $l$  perpendicular to  $(110)$ ).

and fourth, the fourth on the third and fifth, and so on; this is often described as 'repeated twinning.'

In many cases re-entrant angles, &c., betray the existence of such a series of twins; sometimes, however, it is difficult to distinguish such a collection from a single crystal of a symmetry different from that of the individuals composing the series †.

§ 10. We have in chapters ii and x and in the present chapter given a short account of some of the more important properties which observation shows to be possessed by crystals. We shall devote the remainder of this book to a description of the theories suggested to account for these properties. Though these theories contain a certain speculative element, it must be borne in mind that they involve much that is capable of mathematical proof; and that, even if future experiment should show that some of the hypotheses laid down are untenable, yet the usefulness of the theories is not thereby destroyed, even though modification of their details may be necessary.

\* Each crystal of the twin is bounded by faces of the form  $\{111\}$ ; see p. 93, l. 12.

† We get thus so-called 'mimetic twinning.'

## PART II

## CHAPTER XII

## THE STRUCTURE-THEORY.

§ 1. It is now generally assumed that *matter is not continuous but coarse-grained*\*, i. e. that matter is composed of atoms which are practically indivisible and are situated at very small but not infinitesimal distances apart. In a solid these atoms may be grouped into 'molecules'† (either chemical molecules or collections of these) which act as units in building up the solid.

Since the properties of a substance in the neighbourhood of a point within it are not in general affected by the position of the boundary of the substance, we can, and in future shall, consider the substance to be infinitely extended in every direction.

We must now modify our previous definition of homogeneity; we shall say that a substance is homogeneous if, when we take *any* two points *A* and *B* in it, a point *C* can always be found such that the physical behaviour of the substance about *C* is similar to its behaviour about *A*, and such that *B* and *C* (though not necessarily coincident) are so close together† that we cannot distinguish them even by the most refined methods of observation at our disposal.

It may happen that the geometrical positions of *A* and *C* with regard to the collection of molecules are always similar; in this case the substance is said to be crystalline. If, however, the physical properties of the substance about *A* and *C*, but *not* its geometrical properties, are always similar the substance is said to be amorphous.

These definitions imply the existence of a connexion between the orientation of various molecules building up a crystalline medium, which does not exist in the case of an amorphous (or 'vitreous') material.

On the fundamental assumptions of the coarse-grainedness of matter and on our definition of a crystalline medium we

\* See the Presidential address, British Association, 1901.

† A few millionths of a millimetre apart perhaps.

can build up a theory of the internal structure of crystals; or to speak more correctly, we can lay down certain rules to which all such theories must conform.

§ 2. If the collection of molecules composing a homogeneous crystalline medium (of infinite extent) has\* a symmetry-operation it must possess\* an infinite number of symmetry-operations; e. g. if it has an  $n$ -al rotation-axis it must have an infinite number of parallel rotation-axes—this follows from the definitions of homogeneity and crystalline matter in § 1. All the symmetry-operations of such a collection form a group† which contains an infinite number of operations; it is an ‘infinite group.’

We shall now determine (chapters xiii, xiv, xvi–xxiii) the properties of all possible distinct groups of this sort; they are 230 in number as we shall prove later. Each group is completely defined when we have given the relative positions of all its elements of symmetry.

\* A figure may be said to ‘have’ or ‘possess’ a symmetry-operation when it is brought to self-coincidence by that operation.

† This was proved on p. 47.

## CHAPTER XIII

## LATTICES AND TRANSLATIONS.

§ 1. A series of points on a straight line such that the distance between any two consecutive points of the series is constant and equal to  $a$  is called a 'regular row,' or more simply a *row* of points of interval  $a$  (Fig. 64).

A series of coplanar parallel lines such that the distance between any two consecutive lines of the series is constant and equal to  $a$  will be called a *set* of lines of interval  $a$ .

The sum total of the points of intersection of any two coplanar sets of lines is called a 'regular plane net,' or more simply a *net* (Fig. 65).

A series of parallel planes such that the distance between any two consecutive planes of the series is constant and equal to  $a$  will be called a *set* of planes of interval  $a$ .

The sum total of the points of intersection of any three sets of planes is called a 'regular space-lattice,' or more simply a *lattice*\* (Fig. 66).

§ 2. The group of movements which brings the collection of molecules composing a solid homogeneous crystalline material to self-coincidence cannot contain an infinitesimal translation, for no two of the molecules are at an infinitesimal distance apart. Suppose that the *smallest* translation (if any) which the group possesses is a finite † translation  $T$  ‡ represented in magnitude and direction by  $OA_1$  (Fig. 64). Take

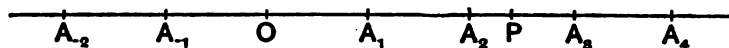


Fig. 64.

$A_1A_2 = A_2A_3 = \dots = A_{-1}O = A_{-2}A_{-1} = \dots = OA_1$ ; then  $T$  can equally well be represented by  $A_1A_2$ ,  $A_2A_3$ ,  $-OA_{-1}$ ,  $-A_1O$ , or in fact by *any* straight line in the direction of and equal to  $OA_1$ .

\* See note at the end of this chapter on p. 127.

† That is, 'neither infinite nor infinitesimal,' and so throughout this book.

‡ The group may have other translations equal to  $T$  but has none smaller than it.

The group must possess translations represented by  $OA_2, OA_3, \dots, OA_{-1}, OA_{-2}, \dots$ ; for since  $T$  is an operation of the group so are  $T^2, T^3, \dots, T^{-1}, T^{-2}, \dots$  (p. 45). Moreover there is no translation of the group parallel to  $OA_1$  not included in this series; for if there were such a translation  $T_1$ , represented by  $OP$  where  $P$  lay between  $A_n$  and  $A_{n+1}$ \*, then  $T_1 \cdot T^{-n}$  would be a translation of the group represented by  $A_nP$ . This would be contrary to our hypothesis that  $T$  is the smallest translation of the group, since  $A_nP < OA_1$ .

If we represent  $OA_1$  by  $2\tau_1$ †;  $OA_2, OA_3, \dots, OA_{-1}, OA_{-2}, \dots$  are represented by  $4\tau_1, 6\tau_1, \dots, -2\tau_1, -4\tau_1, \dots$ .

It is often convenient to denote a translation by the same symbol as the line representing it; we talk of 'the translations  $2\tau_1, 4\tau_1, 6\tau_1, \dots, -2\tau_1, -4\tau_1, \dots$ ' instead of 'the translations  $T, T^2, T^3, \dots, T^{-1}, T^{-2}, \dots$ '.

Such a translation as  $2\tau_1$  or  $-2\tau_1$ , which is the smallest translation of the group in its own direction, is called a *primitive* translation in that direction.

§ 3. Now suppose that the group contains a translation  $T'$  whose direction is different from that of  $T$ , and which is not greater than any translation of the group not parallel to  $T$ ‡. Let  $T'$  be represented in magnitude and direction by  $OB_1 = 2\tau_2$ . Along  $OB_1$  take a 'row' of points  $\dots B_{-2}, B_{-1}, O, B_1, B_2, B_3, \dots$  of interval  $OB_1$ , through them draw parallels to  $OA_1$ , and through  $\dots A_{-2}, A_{-1}, O, A_1, A_2, A_3, \dots$  draw parallels to  $OB_1$ . These two 'sets' of lines meet at the points of a net (Fig. 65).

Let  $E$  be any point of this net, then  $OE$  represents a translation of the group; for if  $E$  be the intersection of the line through  $A_m$  parallel to  $OB_1$ , and the line through  $B_n$  parallel to  $OA_1$ ,  $OE$  represents the resultant of the translations  $OA_m$  and  $A_mE$ , i.e. of  $OA_m$  and  $OB_n$  (since two parallel and equal lines represent the same translation), i.e. of  $2m\tau_1$  and  $2n\tau_2$ .

There are no translations of the group parallel to the plane  $OA_1B_1$  other than those represented by lines such as  $OE$ . For suppose there could be such a translation  $OV$ §, where  $V$  lies in (or on the perimeter of ||) the parallelogram  $p$  formed

\* If the point  $A_n$  is supposed to coincide with  $O$ , this will cover all possible cases;  $n$  being zero or any positive or negative integer.

† The 2 is inserted to avoid fractions later.

‡  $T'$  may be equal to  $T$ , but must not be smaller.

§ 'The translation  $OV$ ' means 'the translation represented in magnitude and direction by  $OV$ ,' and so throughout the book.

|| But not at a vertex of  $p$ .

by the lines through  $A_{m-1}$  and  $A_m$  parallel to  $OB_1$  and the lines through  $B_{n-1}$  and  $B_n$  parallel to  $OA_1$ . Then combining the translation  $OV$  with the translations  $T^{1-m}$  and  $T^{1-n}$  we see that there must be a point  $F$  situated with respect to  $OA_1DB_1$  in the same way that  $V$  is situated with respect

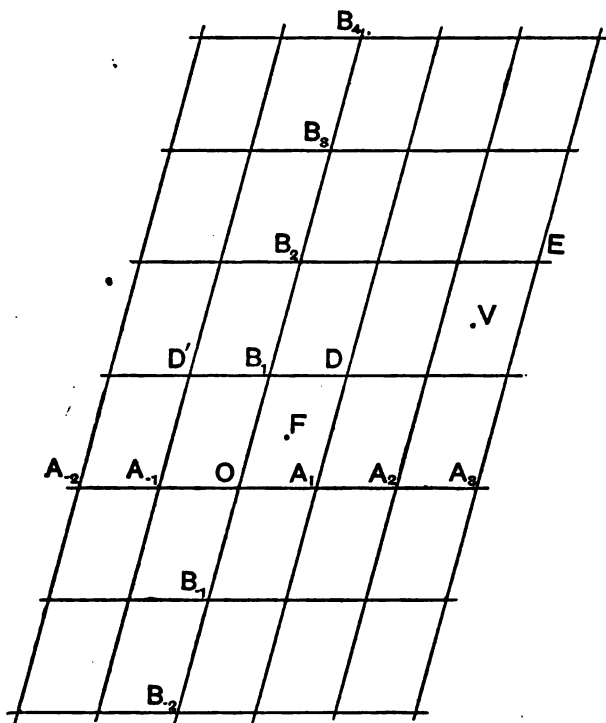


Fig. 65.

to  $p$ , and such that  $OF$  is a translation of the group. In this case the lines  $FA_1$ ,  $FD$ ,  $FB_1$  would also represent translations of the group\*, and would therefore be  $\angle OB_1$  (and hence  $\angle OA_1$ ). Consequently the angles subtended at  $F$  by all the sides of  $OA_1DB_1$  would be  $\gt \frac{\pi}{3}$ . This however is impossible, for their sum is  $2\pi$ .

If  $E$  is the intersection of a line through  $A_m$  parallel to  $OB_1$ , and a line through  $B_n$  parallel to  $OA_1$ ,  $OE$  represents the resultant of the translation  $OA_m$  followed by the transla-

\* For  $FA_1$  would represent the resultant of the translations  $FO$ ,  $OA_1$ , &c.

tion  $A_mE$ , i. e. of  $2m\tau_1$  followed by  $2n\tau_2$ . We shall write this resultant in the form  $2m\tau_1 + 2n\tau_2$ . Similarly  $OE$  represents the resultant of the translation  $OB_n$  followed by the translation  $B_nE$ , i. e. of  $2n\tau_2$  followed by  $2m\tau_1$ . We write this resultant in the form  $2n\tau_2 + 2m\tau_1$ . Then  $2m\tau_1 + 2n\tau_2 = 2n\tau_2 + 2m\tau_1$ , and hence the two translations  $2m\tau_1$  and  $2n\tau_2$  are permutable operations. An exactly similar proof leads us to the theorem:—

*Any two translations are permutable operations.*

We choose the addition sign instead of the multiplication to represent the resultant of two translations (if the translations are denoted by the same symbol as the lines representing them), in order to be consistent with the fact that the translation  $2p\tau_1$  followed by the translation  $2q\tau_1$  is evidently equivalent to the translation  $2(p+q)\tau_1$ .

We have proved that *any* translation of the group parallel to the plane  $OA_1B_1$  can be represented by  $2m\tau_1 + 2n\tau_2$  where  $m$  and  $n$  are positive or negative integers\*; we therefore call  $2\tau_1$  and  $2\tau_2$  a *primitive pair of translations* 'in' or 'parallel to the plane'  $OA_1B_1$ .  $2\tau_1$  and  $2\tau_2$  are not the only primitive pair of translations parallel to this plane; for let  $2\tau'_1, 2\tau'_2$  be any two translations of the group parallel to the plane, then

$$\begin{aligned} 2\tau'_1 &= 2p_1\tau_1 + 2p_2\tau_2 \\ 2\tau'_2 &= 2q_1\tau_1 + 2q_2\tau_2 \end{aligned}$$

where  $p_1, p_2, q_1, q_2$  are integers.

Solving † we have

$$\begin{aligned} \tau_1 &= \frac{q_2}{l}\tau'_1 - \frac{p_2}{l}\tau'_2 \\ \tau_2 &= -\frac{q_1}{l}\tau'_1 + \frac{p_1}{l}\tau'_2 \end{aligned}$$

where  $l \equiv \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix}$  and is therefore integral.

If now  $l = \pm 1$  (and by the theory of continued fractions it is possible to find an infinite number of integral solutions of the equation  $p_1q_2 - p_2q_1 = \pm 1$ )  $2\tau_1$  and  $2\tau_2$  are expressed in the form  $2m\tau'_1 + 2n\tau'_2$  ( $m$  and  $n$  integral); but by hypothesis any translation of the group parallel to the plane  $OA_1B_1$  can

\* Including zero.

† The justification of this step lies in the fact that  $p\tau + q\tau = (p+q)\tau$  where  $\tau$  is any translation, and in the fact that all translations are permutable.

be expressed in the form  $2m\tau_1 + 2n\tau_2$ , and therefore any translation parallel to this plane can be expressed in the form  $2m\tau_1' + 2n\tau_2'$ ; i. e.  $2\tau_1', 2\tau_2'$  are a primitive pair.

Conversely, if  $2\tau_1', 2\tau_2'$  are a primitive pair,  $l = \pm 1$ . For in this case  $l$  must be a divisor of  $p_1, p_2, q_1, q_2$  (otherwise  $2\tau_1$  and  $2\tau_2$  could not be expressed in the form  $2m\tau_1' + 2n\tau_2'$ ). Let therefore  $p_1 = p_1'l, p_2 = p_2'l, q_1 = q_1'l, q_2 = q_2'l$ , where  $p_1', p_2', q_1', q_2', l$  are integers; then

$$l = \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} = l^3 \times \begin{vmatrix} p_1' & p_2' \\ q_1' & q_2' \end{vmatrix};$$

and therefore  $1 = l \times \begin{vmatrix} p_1' & p_2' \\ q_1' & q_2' \end{vmatrix} = l \times \text{an integer.}$

$$\therefore l = \pm 1.$$

The above reasoning holds if  $2\tau_1, 2\tau_2$  represent *any* primitive pair (not necessarily  $OA_1, OB_1$ ).

Since by the theory of continued fractions integers  $p_1, p_2$  can be found to make  $p_1q_2 - p_2q_1 = \pm 1$  if  $q_1, q_2$  are *any* given integers having no common factor; therefore in any net we can find a pair of primitive translations of which one is represented by the line joining  $O$  to *any given point* of the net  $E$ , provided no other point of the net lies on the finite straight line  $OE$ .

It is easily verified that  $2\tau_1, 2m\tau_1 \pm 2\tau_2$  (where  $m$  is any integer, and  $2\tau_1, 2\tau_2$  are *any* primitive pair) form a primitive pair; in particular  $OA_1, B_1A_1$  ( $2\tau_1, 2\tau_1 - 2\tau_2$ ) of Fig. 65 are such a pair. Hence if one pair of sides of a triangle represents a primitive pair of translations, so also does any pair of sides of the triangle.

§ 4. To find the area of the triangle whose vertices are any three points of the net of Fig. 65.

We may take one of the points as our origin of reference  $O$ ; let  $a$  and  $b$  be the others.

Suppose  $Oa = 2p_1\tau_1 + 2p_2\tau_2, Ob = 2q_1\tau_1 + 2q_2\tau_2$ , where  $OA_1 = 2\tau_1, OB_1 = 2\tau_2$  represent a primitive pair of translations in the plane  $OA_1B_1$ ; and denote the angle  $A_1OB_1$  by  $\delta$ .

Let the coordinates of  $A_1, B_1$  referred to any two rectangular axes through  $O$  in the plane  $OA_1B_1$  be  $(x_1, y_1), (x_2, y_2)$ ; then the coordinates of  $a, b$  are

$$(p_1x_1 + p_2x_2, p_1y_1 + p_2y_2), (q_1x_1 + q_2x_2, q_1y_1 + q_2y_2),$$

respectively.

Then the area of  $Oab$

$$\begin{aligned}
 &= \frac{1}{2} \begin{vmatrix} 1 & p_1x_1 + p_2x_2 & p_1y_1 + p_2y_2 \\ 1 & q_1x_1 + q_2x_2 & q_1y_1 + q_2y_2 \\ 1 & 0 & 0 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} \times \begin{vmatrix} 1 & x_1 & x_2 \\ 1 & y_1 & y_2 \\ 1 & 0 & 0 \end{vmatrix} \\
 &= \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} \times \text{the area of the triangle } OA_1B_1 \\
 &= \frac{1}{2} \begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} OA_1 \cdot OB_1 \cdot \sin \delta.
 \end{aligned}$$

$Oa, Ob$  represent a primitive pair of translations if and only if

$$\begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} = \pm 1;$$

hence: *the areas of all triangles two of whose sides represent a primitive pair of translations in a given plane are equal: and conversely, if the area of a triangle  $D$  whose vertices are points of the net representing the translations in any plane is equal to that of a triangle, two of whose sides represent a primitive pair of translations in that plane, any two sides of  $D$  also represent a primitive pair.*

The above converse gives us at once another proof of the theorem that 'if one pair of sides of a triangle represents a primitive pair of translations, so also does any pair of sides of the triangle.'

§ 5. Let  $OC_1 = 2\tau_3$  be the smallest of the translations of the group (if any) which is not parallel to the plane  $OA_1B_1$ . Along  $OC_1$  construct a row of points  $\dots C_{-2}, C_{-1}, O, C_1, C_2, C_3, \dots$  of interval  $OC_1$ , and through them draw planes parallel to  $OA_1B_1$ ; through  $\dots A_{-2}, A_{-1}, O, A_1, A_2, A_3, \dots$  draw planes parallel to  $OB_1C_1$ , and through  $\dots B_{-2}, B_{-1}, O, B_1, B_2, B_3, \dots$  draw planes parallel to  $OC_1A_1$ . Then if  $E$  be any point of the lattice formed by the intersections of these three sets of planes,  $OE$  is a translation of the group\* (Fig. 66), and is included among the translations  $2m\tau_1 + 2n\tau_2 + 2s\tau_3$ † (where  $m, n, s$  are integers). There are no other translations of the group; for if there were such a translation  $OV$  where  $V$  is not a point of the lattice, there must also be a translation of the group represented by  $OF$ , where  $F$  lies in or on the parallelepipedon, three of whose adjacent edges are  $OA_1, OB_1, OC_1$ .\*

\* The proof is exactly similar to that of the corresponding theorem of § 3.

† Now  $2\tau_1 = OA_1, 2\tau_2 = OB_1$ .

In that case the lines joining  $F$  to all the vertices of this parallelepipedon would represent translations and would therefore be  $\sphericalangle OC_1$ ; and consequently the angles subtended at  $F$  by all the sides of the parallelepipedon would be  $\sphericalangle \frac{\pi}{3}^*$ .

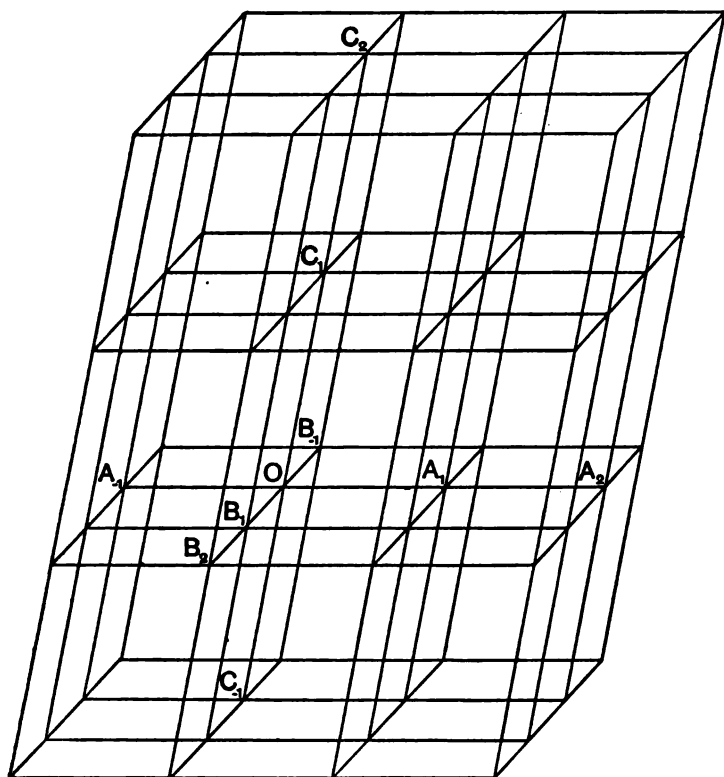


Fig. 66.

This however is impossible; for let the planes through  $F$  and the sides cut a sphere whose centre is  $F$  in the twelve edges of six foursided polygons covering the sphere. The area of at least one of these polygons must be  $\sphericalangle$  a sixth of the area of the surface of the sphere, that is,  $\sphericalangle$  the area of a regular foursided polygon whose sides are  $\cos^{-1} \frac{1}{2}$ . But the area of a foursided

\* Cf. p. 117.

polygon whose sides are all  $\succ \frac{\pi}{3}$  ( $= \cos^{-1} \frac{1}{2}$ ) is evidently  $<$  the area of some four-sided polygon whose sides are  $\cos^{-1} \frac{1}{2}$ \*, and is therefore  $<$  the area of the regular four-sided polygon whose sides are  $\cos^{-1} \frac{1}{2}$ †. Hence at least one of the angles subtended by the sides of the parallelepipedon  $F$  must be  $> \frac{\pi}{3}$ .

We see then that every translation of a group (which contains only finite translations) can be expressed in the form  $2m\tau_1 + 2n\tau_2 + 2s\tau_3$  where  $m, n, s$  are integers;  $2\tau_1, 2\tau_2, 2\tau_3$  are therefore said to form a *primitive triplet*‡.

There are an infinite number of primitive triplets; for let  $2\tau_1', 2\tau_2', 2\tau_3'$  be any three translations of the group

$$\begin{aligned} \text{then} \quad 2\tau_1' &= 2p_1\tau_1 + 2p_2\tau_2 + 2p_3\tau_3 \\ 2\tau_2' &= 2q_1\tau_1 + 2q_2\tau_2 + 2q_3\tau_3 \\ 2\tau_3' &= 2r_1\tau_1 + 2r_2\tau_2 + 2r_3\tau_3 \end{aligned}$$

where  $p, q, r$ , &c., are integers.

Solving, for  $\tau_1, \tau_2, \tau_3$

$$\begin{aligned} \text{we have} \quad 2l\tau_1 &= 2P_1\tau_1' + 2Q_1\tau_2' + 2R_1\tau_3' \\ 2l\tau_2 &= 2P_2\tau_1' + 2Q_2\tau_2' + 2R_2\tau_3' \\ 2l\tau_3 &= 2P_3\tau_1' + 2Q_3\tau_2' + 2R_3\tau_3' \end{aligned}$$

$$\text{where } l \equiv \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix}, \text{ and } P_1, Q_1, R_1, \text{ \&c., are the } c$$

factors of  $p_1, q_1, r_1$ , &c., in this determinant.

Now if  $l = \pm 1$  (and we can determine  $p_1, p_2, p_3, q_1, q_2, q_3, r_1, r_2, r_3$  in an infinite number of ways so as to satisfy the condition),  $2\tau_1', 2\tau_2', 2\tau_3'$  are a primitive triplet; for since any translation can be expressed in the form  $2m\tau_1 + 2n\tau_2 + 2s\tau_3$  it can, if  $l = \pm 1$ , be expressed in the form  $2m\tau_1' + 2n\tau_2' + 2s\tau_3'$ . Conversely if  $2\tau_1', 2\tau_2', 2\tau_3'$  are a primitive triplet  $l = \pm 1$ . For since in this case  $l$  divides  $P_1, Q_1, R_1, P_2, Q_2, R_2, P_3, Q_3, R_3$

\* Let  $EFGH$  be a polygon whose sides are  $\succ \frac{\pi}{3}$ ; then the area of polygon  $EF'GH'$  (where  $EF' = GF' = EH' = CH' = \cos^{-1} \frac{1}{2}$ ) is  $>$  that of polygon  $EF'GH'$  (where  $EF' = GF' = \frac{1}{2}(EF + GF)$  and  $EH' = GH' = \frac{1}{2}(EH + GH)$  and is therefore  $>$  that of  $EFGH$ .

† This is a particular case of the theorem that 'the area of a spherical polygon, whose sides are given, is a maximum when the vertices all lie on a small circle.'

‡ German—ein primitives Tripel.

Let  $P_1 = P_1' l$ ,  $P_2 = P_2' l$ ,  $P_3 = P_3' l$ , &c., where  $l$ ,  $P_1'$ ,  $Q_1'$ ,  $R_1'$ , &c., are integers.

Then  $l^2 = \begin{vmatrix} P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \\ P_3 & Q_3 & R_3 \end{vmatrix} = l^3 \times \begin{vmatrix} P_1' & Q_1' & R_1' \\ P_2' & Q_2' & R_2' \\ P_3' & Q_3' & R_3' \end{vmatrix},$

and  $\therefore 1 = l \times \begin{vmatrix} P_1' & Q_1' & R_1' \\ P_2' & Q_2' & R_2' \\ P_3' & Q_3' & R_3' \end{vmatrix} = l \times \text{an integer},$

and  $\therefore l = \pm 1.$

The above reasoning holds if  $2\tau_1, 2\tau_2, 2\tau_3$  are any primitive triplet (not necessarily  $OA_1, OB_1, OC_1$ ).

We may easily verify that  $2\tau_1, 2\tau_2, 2\tau_3 + 2m_1\tau_1 + 2m_2\tau_2$  are a primitive triplet (where  $m_1, m_2$  are any integers, and  $\tau_1, 2\tau_2, 2\tau_3$  any primitive triplet); in particular we may easily verify that any three sides of the tetrahedron  $OA_1B_1C_1$  (Fig. 5) form a primitive triplet; hence:—*If any three non-coplanar sides of a tetrahedron represent a primitive triplet of a group of translations; the same is true of any other three non-coplanar sides of the tetrahedron.*

§ 6. To find the conditions that any two translations both parallel to a plane  $h$  may be a primitive pair parallel to  $h$ .

We shall suppose any translation represented by a line through the origin  $O^*$ . Let then the two translations be  $q_1\tau_1 + 2q_2\tau_2 + 2q_3\tau_3 \equiv 2\tau_2', 2r_1\tau_1 + 2r_2\tau_2 + 2r_3\tau_3 \equiv 2\tau_3'$ , where  $\tau_1, 2\tau_2, 2\tau_3$  are any primitive triplet. Then if any other translation parallel to  $h$  is  $2m_1\tau_1 + 2m_2\tau_2 + 2m_3\tau_3 \equiv 2\tau_1'$ , we must have

$$\left. \begin{aligned} m_1 &= \lambda q_1 + \mu r_1 \\ m_2 &= \lambda q_2 + \mu r_2 \\ m_3 &= \lambda q_3 + \mu r_3 \end{aligned} \right\}, \text{ since } \begin{vmatrix} m_1 & q_1 & r_1 \\ m_2 & q_2 & r_2 \\ m_3 & q_3 & r_3 \end{vmatrix} = 0$$

$2\tau_1', 2\tau_2', 2\tau_3'$  are coplanar.

Solving, we have

$$\lambda = \frac{m_2 r_3 - m_3 r_2}{q_2 r_3 - q_3 r_2} = \frac{m_3 r_1 - m_1 r_3}{q_3 r_1 - q_1 r_3} = \frac{m_1 r_2 - m_2 r_1}{q_1 r_2 - q_2 r_1}.$$

$q_2 r_3 - q_3 r_2, q_3 r_1 - q_1 r_3, q_1 r_2 - q_2 r_1$  have no common factor but unity, these equations can only hold good when  $\lambda$  is integral.

Similarly  $\mu$  must be integral. Therefore any translation  $2\tau_1'$  in the plane of  $2\tau_2', 2\tau_3'$  can be expressed in the form

\* This is lawful, for all lines of the same length and direction represent the same translation.

$2\lambda\tau_3' + 2\mu\tau_3'$  where  $\lambda$  and  $\mu$  are integers; i. e.  $2\tau_2'$  and  $2\tau_3'$  form a primitive pair parallel to the plane  $h$ .

Now suppose  $q_2r_3 - q_3r_2$ ,  $q_3r_1 - q_1r_3$ ,  $q_1r_2 - q_2r_1$  have a prime common factor  $d$ . If  $q_1, q_2, q_3$  have a common factor  $2\tau_2'$  is not a primitive translation in its own direction and therefore evidently  $2\tau_2'$ ,  $2\tau_3'$  cannot form a primitive pair parallel to  $h^*$ . Suppose  $q_1, q_2, q_3$  have no common factor, then one of them, say  $q_1$ , is prime to  $d$ .

Find an integer  $L$  such that  $Lq_1 + Mr_1$  is divisible by  $d$ ,  $M$  being any integer prime to  $d$ †.

Then  $q_1(Lq_2 + Mr_2) = q_2(Lq_1 + Mr_1) + (q_1r_2 - q_2r_1)M$  and  $q_1(Lq_3 + Mr_3) = q_3(Lq_1 + Mr_1) + (q_1r_3 - q_3r_1)M$  are both divisible by  $d$ ; and therefore  $Lq_2 + Mr_2$ ,  $Lq_3 + Mr_3$  are both divisible by  $d$ , for  $q_1$  is prime to  $d$ .

Put  $L = \lambda d$ ,  $M = \mu d$ . Then we see that quantities  $\lambda, \mu$  can be chosen, such that one at least is not an integer‡, and such that  $\lambda q_1 + \mu r_1$ ,  $\lambda q_2 + \mu r_2$ ,  $\lambda q_3 + \mu r_3$  are integers. In this case the translation  $2\lambda\tau_2' + 2\mu\tau_3'$  belongs to the group, and it is evidently parallel to the plane  $h$ .

But  $\lambda$  and  $\mu$  are not both integral, and therefore  $2\tau_2'$ ,  $2\tau_3'$  are not a primitive pair. Hence:—

*The translations  $2q_1r_1 + 2q_2r_2 + 2q_3r_3$ ,  $2r_1r_1 + 2r_2r_2 + 2r_3r_3$  do or do not form a primitive pair in the plane in which they lie according as  $q_2r_3 - q_3r_2$ ,  $q_3r_1 - q_1r_3$ ,  $q_1r_2 - q_2r_1$  have not or have a common factor ||.*

If  $2\tau_2'$ ,  $2\tau_3'$  are a primitive pair in their own plane then there is a translation  $2\tau$  of the group, such that  $2\tau$ ,  $2\tau_2'$ ,  $2\tau_3'$  form a primitive triplet.

For by the theory of continued fractions it is always possible to find integers  $p_1, p_2, p_3$  such that

$$\begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} = p_1(q_2r_3 - q_3r_2) + p_2(q_3r_1 - q_1r_3) + p_3(q_1r_2 - q_2r_1) = 1$$

if  $q_2r_3 - q_3r_2$ ,  $q_3r_1 - q_1r_3$ ,  $q_1r_2 - q_2r_1$  have no common factor.

§ 7. To find the volume of a tetrahedron whose vertices are any four points of the lattice of Fig. 66.

\* See p. 119.

† This is always possible, for  $q_1, 2q_1, 3q_1, \dots, (d-1)q_1$  leave different remainders when divided by  $d$ , since  $q_1$  is prime to  $d$ .

‡  $\mu$  is not an integer, for  $M$  was taken prime to  $d$ .

|| This proof is due to Prof. E. B. Elliott. It may be noted that if  $q_1r_3 - q_3r_1$  and  $q_2r_1 - q_1r_2$  are both divisible by  $d$  which is prime to  $q_1$ , then of necessity  $q_3r_2 - q_2r_3$  is divisible by  $d$ , for  $q_1(q_2r_3 - q_3r_2) + q_2(q_3r_1 - q_1r_2) + q_3(q_1r_2 - q_2r_1) \equiv 0$ .

We may take one of the points as our origin of reference  $O$ ;   
 et  $a, b$ , and  $c$  be the other three.

$$\begin{aligned} \text{Suppose } Oa &= 2p_1\tau_1 + 2p_2\tau_2 + 2p_3\tau_3 \\ Ob &= 2q_1\tau_1 + 2q_2\tau_2 + 2q_3\tau_3 \\ Oc &= 2r_1\tau_1 + 2r_2\tau_2 + 2r_3\tau_3 \end{aligned}$$

where  $OA_1 = 2\tau_1$ ,  $OB_1 = 2\tau_2$ ,  $OC_1 = 2\tau_3$  represent any primitive triplet of translations; and let  $OA_1$ ,  $OB_1$ ,  $OC_1$  make angles  $\alpha, \beta, \gamma$  with one another.

Let the coordinates of  $A_1, B_1, C_1$  referred to any three orthogonal axes through  $O$  be  $(x_1, y_1, z_1)$  or  $(OA_1, l_1, OA_1, m_1, OA_1, n_1)$ ;   
 $(x_2, y_2, z_2)$  or  $(OB_1, l_2, OB_1, m_2, OB_1, n_2)$ ; and  $(x_3, y_3, z_3)$  or  $(OC_1, l_3, OC_1, m_3, OC_1, n_3)$  respectively. Then the coordinates of  $a, b, c$  are

$$\begin{aligned} (p_1x_1 + p_2x_2 + p_3x_3, p_1y_1 + p_2y_2 + p_3y_3, p_1z_1 + p_2z_2 + p_3z_3), \\ (q_1x_1 + q_2x_2 + q_3x_3, q_1y_1 + q_2y_2 + q_3y_3, q_1z_1 + q_2z_2 + q_3z_3), \\ (r_1x_1 + r_2x_2 + r_3x_3, r_1y_1 + r_2y_2 + r_3y_3, r_1z_1 + r_2z_2 + r_3z_3) \end{aligned}$$

respectively.

Therefore the volume of the tetrahedron  $Oabc$

$$\begin{aligned} &= \frac{1}{6} \times \begin{vmatrix} 1 & 1 & 1 \\ 0 & p_1x_1 + p_2x_2 + p_3x_3 & p_1y_1 + p_2y_2 + p_3y_3 & p_1z_1 + p_2z_2 + p_3z_3 \\ 0 & q_1x_1 + q_2x_2 + q_3x_3 & q_1y_1 + q_2y_2 + q_3y_3 & q_1z_1 + q_2z_2 + q_3z_3 \\ 0 & r_1x_1 + r_2x_2 + r_3x_3 & r_1y_1 + r_2y_2 + r_3y_3 & r_1z_1 + r_2z_2 + r_3z_3 \end{vmatrix} \\ &= \frac{1}{6} \times \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} \times \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} \\ &= \frac{OA_1 \cdot OB_1 \cdot OC_1}{6} \times \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} \times \begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} \\ &= \frac{OA_1 \cdot OB_1 \cdot OC_1}{6} \times \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} \\ &\quad \times \left\{ \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \right\}^{\frac{1}{2}} \\ &= \frac{OA_1 \cdot OB_1 \cdot OC_1}{6} \times \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} \\ &\quad \times \left\{ \begin{vmatrix} l_1^2 + m_1^2 + n_1^2 & l_1l_2 + m_1m_2 + n_1n_2 & l_1l_3 + m_1m_3 + n_1n_3 \\ l_1l_2 + m_1m_2 + n_1n_2 & l_2^2 + m_2^2 + n_2^2 & l_2l_3 + m_2m_3 + n_2n_3 \\ l_1l_3 + m_1m_3 + n_1n_3 & l_2l_3 + m_2m_3 + n_2n_3 & l_3^2 + m_3^2 + n_3^2 \end{vmatrix} \right\}^{\frac{1}{2}} \end{aligned}$$

$$= \frac{OA_1 \cdot OB_1 \cdot OC_1}{6} \times \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} \times \left\{ \begin{vmatrix} 1 & \cos \gamma & \cos \beta \\ \cos \gamma & 1 & \cos \alpha \\ \cos \beta & \cos \alpha & 1 \end{vmatrix} \right\}^{\frac{1}{2}}.$$

Now we note that if through any point  $P$  lines  $PA$ ,  $PB$ ,  $PC$  are drawn equal and parallel to any three non-coplanar edges (e.g.  $A_1C_1$ ,  $OB_1$ ,  $OC_1$ ) of a tetrahedron  $OA_1B_1C_1$ , then the tetrahedra  $PABC$ ,  $OA_1B_1C_1$  are equal; for they stand on equal bases ( $PBC$ ,  $OB_1C_1$ ) and are evidently of equal altitude.

Therefore, since  $Oa$ ,  $Ob$ ,  $Oc$  represent a primitive triplet, if and

only if  $\begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{vmatrix} = \pm 1$ , we have the following theorem:—

*The volumes of all tetrahedra, three of whose non-coplanar sides represent a primitive triplet of translations, are equal: and conversely, if the volume of a tetrahedron  $D$ , whose vertices are points of the lattice representing the group of translations, is equal to that of a tetrahedron three of whose non-coplanar sides represent a primitive triplet; any three non-coplanar sides of  $D$  also represent a primitive triplet.*

The above converse gives us at once another proof of the theorem that 'if any three non-coplanar sides of a tetrahedron represent a primitive triplet, the same is true of any other three non-coplanar sides of the tetrahedron.'

§ 8. If  $O$ ,  $a$ ,  $b$ ,  $c$  be any points of a lattice such that  $Oa$ ,  $Ob$ ,  $Oc$  represent a primitive triplet, then  $Ob$ ,  $Oc$  must represent a primitive pair of translations parallel to the plane  $Obc$ . For if they do not, let  $Ob'$ ,  $Oc'$  represent a primitive pair in that plane. Then the triangle  $Ob'c' < Obc$ , and therefore the tetrahedron  $AOb'c' < AObc$  which is impossible.

A similar result holds for  $Oc$ ,  $Oa$ , and for  $Oa$ ,  $Ob$ .

Again  $Oa$  must be a primitive translation in its own direction. For if it is not let  $Oa'$  be the primitive translation in that direction. Then  $Oa' < Oa$  and therefore the tetrahedron  $a'Obc < aObc$  which is impossible.

A similar result holds for  $Ob$  and  $Oc$ .

§ 9. In § 5 we arrived at a primitive triplet by taking  $A_1$  as near  $O$  as possible;  $B_1$  as near  $O$  as is consistent with the distinctness of the lines  $OA_1$ ,  $OB_1$ ; and  $C_1$  as near  $O$  as is possible when points in the plane  $OA_1B_1$  are excluded. We proved then that  $OA_1$ ,  $OB_1$ ,  $OC_1$  form a primitive triplet.

We can also arrive at a primitive triplet as follows:—Take

any point  $A$  of the lattice such that  $OA$  represents a primitive translation in its own direction. In any net whose plane passes through  $OA$  take a point  $B$  which is no farther from  $OA$  than any other point of this net not lying in  $OA$ . Then  $OA, OB$  are a primitive pair in their own plane; for it is possible to find some point  $D$  in the net such that  $OA, OD$  are a primitive pair (p. 119), and  $D$  cannot be farther from  $OA$  than  $B$ , for then we should have the triangle  $ODA > OBA$ .

Now take a point  $C$  of the lattice as near the plane  $OAB$  as possible, but not lying in this plane. Then  $OA, OB, OC$  represent a primitive triplet, for it is possible to find some point  $E$  such that  $OA, OB, OE$  have this property (p. 124), and  $E$  cannot be farther from  $OAB$  than  $C$ , for then we should have the tetrahedron  $EOAB > COAB$ .

There is a large amount of choice in this way of finding a primitive triplet; for even when we have fixed  $OA$  and the plane of  $OAB$  we have an infinite number of points which may be taken as  $B$  (lying along two lines parallel to  $OA$ ), and an infinite number of points which may be taken as  $C$  (lying in two planes parallel to  $OAB$ ).

#### NOTE ON TERMINOLOGY.

The following is a comparison of words used by various authors in this subject:—

<i>English.</i>	<i>French.</i>	<i>German.</i>
Row	Rangée	Punktreihe.
Net	Réseau	Punktnetz.
Lattice	Assemblage	Raumgitter.
Point (of net or lattice)	Sommet	Punkt.

The words 'lattice, assemblage, Raumgitter' are not satisfactory; 'assemblage' is too vague, while 'lattice' suggests a *plane* figure. We have, however, employed it as shorter than 'space-lattice,' which is in more general use, but which does not suggest a three-dimensional figure in spite of the prefix 'space.'

## CHAPTER XIV

### SYMMETRICAL LATTICES.

§ 1. We have now proved that every translation of any group can be represented by straight lines drawn from any point of a certain lattice to every other point of the lattice. If we have any two translations of the group

$$T = 2p_1\tau_1 + 2p_2\tau_2 + 2p_3\tau_3, \quad T' = 2q_1\tau_1 + 2q_2\tau_2 + 2q_3\tau_3,$$

then  $T.T' = 2(p_1 + q_1)\tau_1 + 2(p_2 + q_2)\tau_2 + 2(p_3 + q_3)\tau_3$  is also a translation, and belongs to the group; hence the translations of any group form a subgroup.

We shall call this subgroup the *translation-group*. Owing to the fact that the subgroup formed by the translations of any group can be represented by a lattice, it is of importance to determine the various kinds of symmetry possible in a lattice; we shall begin by considering the symmetry of 'nets.'

§ 2. Consider a net which has one of its points  $O$  fixed (Fig. 65); in this case any symmetry-element of the net must be a rotation-axis, axis of rotatory-reflexion, or symmetry-plane passing through  $O$ .

Now evidently every net with one point  $O$  fixed has as symmetry-elements (1) a centre of symmetry at  $O$ , (2) a 2-al rotation-axis through  $O$  perpendicular to the plane of the net, (3) a symmetry-plane coinciding with the plane of the net\*.

In general the net has† no other symmetry-operations leaving  $O$  unmoved, but in particular cases it may possess a 2-al rotation-axis lying in the plane of the net, or a 4-al or 6-al rotation-axis perpendicular to this plane. Evidently a net can possess no symmetry-axis of the first sort other than these‡; if a net has an  $n$ -al symmetry-axis of the second

\* Any two of these symmetry-elements involves the third.

† See footnote, p. 114.

‡ Rotation about any other rotation-axis not perpendicular to the plane of the net does not bring the plane of the net, and therefore does not bring the net to self-coincidence. If there is an  $n$ -al rotation-axis perpendicular to the plane  $n$  may be taken even (for every net has a 2-al rotation-axis perpendicular to its plane). Let a rotation through  $\frac{2\pi}{n}$  about this axis bring  $A_1$  to  $A$ . Then by hypothesis  $A_1A \nless OA_1$ ;  $\therefore n \nless 6$ .

part it must also possess an  $n$ -al or  $2n$ -al rotation-axis owing to the existence of the centre of symmetry, so that a separate discussion is unnecessary. (In particular if the net has a symmetry-plane it has a perpendicular 2-al rotation-axis).

§ 3. Suppose that the net has a 2-al rotation-axis  $l$  (passing through  $O$ ) lying in its own plane; then since the net has a 2-al rotation-axis perpendicular to its plane, by Euler's construction it must have another 2-al rotation-axis  $l'$  lying in its plane perpendicular to  $l$ .

First suppose  $OA_1$  perpendicular to  $OB_1$  in Fig. 65. Then evidently the net has two 2-al rotation-axes lying in its plane, namely, the lines  $OA_1$ ,  $OB_1$ .

In this case the net is said to be *rectangular* (Fig. 67).

Next suppose  $OA_1$  not perpendicular to  $OB_1$ .

First let  $l$  (or  $l'$ ) be perpendicular to  $OA_1$ , and let a rotation through  $\pi$  about  $l$  bring  $OB_1$  into the position  $OB$ , then  $B$  must be a point of the net. Also  $B_1B$  is parallel to  $OA_1$ ,

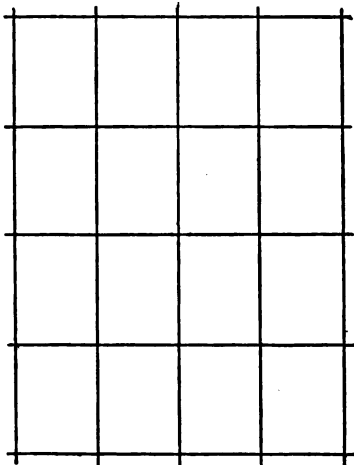


Fig. 67. Rectangular net.

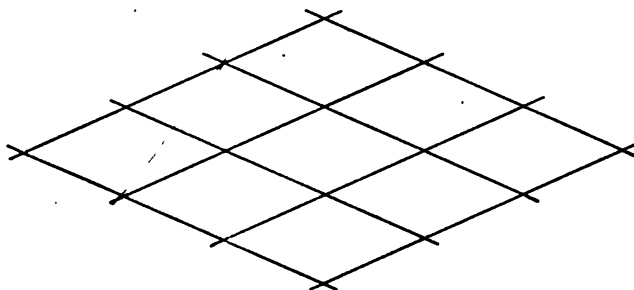


Fig. 68. Rhombic net.

and therefore  $B$  must coincide with one of the points of the net which lie on  $B_1D$  (Fig. 65). But we know that  $OB_1 = OB$  is the shortest distance between two points of the net (unless the two points lie on a line parallel to  $OA_1$ ); therefore  $B$  must

coincide with  $D'$ . Now  $OB_1$ ,  $OD'$  form a primitive pair of translations of the group represented by the net, since the triangles  $OA_1B_1$ ,  $OB_1D'$  are equal (p. 120); hence the group has a primitive pair of *equal* translations. In this case the net is said to be *rhombic* (Fig. 68).

Again, let neither  $l$  nor  $l'$  be perpendicular to  $OA_1$ ; and let  $A_1$  be brought to  $A$  by rotation about  $l$ . Then in our construction of the net of Fig. 65 we can take  $OB_1$  as coinciding with  $OA$ , so that we arrive again at a rhombic set.

§ 4. Suppose that the net has a 4-al rotation-axis  $l$  perpendicular to its plane, and let  $A_1$  be brought to  $A$  by a rotation through  $\frac{\pi}{2}$  about  $l$ .

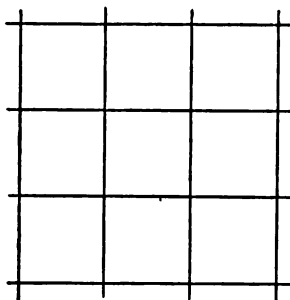


Fig. 69. Square net.

Then we may take  $OB_1$  as coinciding with  $OA$ , and arrive at a net with square meshes which we shall call a *square* net. (Fig. 69).

Suppose that the net has a 6-al rotation-axis  $l$  perpendicular to its plane, and let  $A_1$  be brought to  $A$ ,  $A$  by rotations through  $\frac{2\pi}{3}$  and  $\frac{4\pi}{3}$  about  $l$ . Then we may take  $OB_1$  as coinciding with  $OA$  or  $OA$ , and arrive at a net formed by rhombi with angles  $\frac{\pi}{3}$  and  $\frac{2\pi}{3}$ . Such a net is called *equilateral* (Fig. 70).

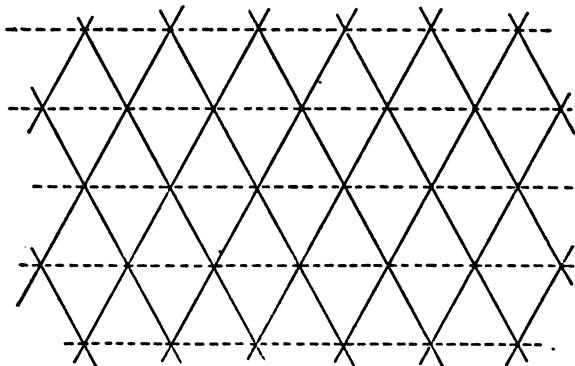


Fig. 70. Equilateral net.

\* We suppose the angle  $A_1OB_1$  acute, as is evidently lawful.

The operations which bring to self-coincidence a rectangular, rhombic, square, or equilateral net, but which leave one of the points unmoved, form the groups  $D_{2h}$ ,  $D_{2h}$ ,  $D_{4h}$ , or  $D_{6h}$  respectively.

§ 5. We now proceed to consider those symmetry-operations of any lattice which leave one point unmoved. As before, it is only necessary to consider operations of the first sort; for every point of a lattice is evidently a centre of symmetry of that lattice. If a lattice has a system of rotation-axes through one of its points, it has a similar and similarly orientated system through each of its points; for any translation of the group represented by the lattice is evidently a symmetry-operation of that lattice, and any symmetry-operation of a figure brings the system formed by its symmetry-elements to self-coincidence (p. 47).

If the symmetry-operations of a lattice which leave one point  $O$  unmoved form a group  $G$ , the lattice will be said to 'have the symmetry of the group  $G$ .'

The perfectly general lattice has no rotation-axes; it has the symmetry of the group  $C_1$ , and is denoted by  $\Gamma_{tr}$ .

Now suppose that the lattice has an  $n$ -al rotation-axis  $l$  through  $O$ , and let  $E_1$  be any point of the lattice. Then if rotations through  $\frac{2\pi}{n}$ ,  $\frac{4\pi}{n}$ ,  $\frac{6\pi}{n}$ , ...,  $\frac{2(n-1)\pi}{n}$  about  $l$  bring  $E_1$  into the positions  $E_2, E_3, E_4, \dots, E_n$ , these points are also points of the lattice which lie in a net perpendicular to  $l$  (Fig. 71). Therefore:—

*Every rotation-axis of a lattice has a net perpendicular to it\*.*

\* The proof fails if  $n = 2$ ; in this case let any points  $E_1, F_1, G_1, \dots$  of the

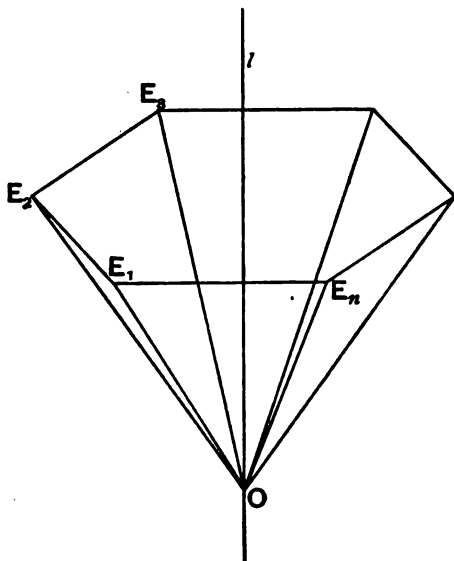


Fig. 71.

Again :—

*Every symmetry-plane of a lattice which contains one point of the lattice, contains a net of points ;*

for the symmetry-plane has a 2-al rotation-axis perpendicular to it, since  $O$  is a centre of symmetry of the lattice.

Since the resultant of all the translations represented by  $OE_1, OE_2, OE_3, \dots, OE_n$  is parallel to  $l$ , therefore :—

*Every rotation-axis of a lattice passing through  $O$  passes through an infinite number of points of the lattice\*.*

If an  $n$ -al rotation-axis  $l$  passes through  $O$  (Fig. 71), a parallel  $n$ -al rotation-axis passes through  $E_1$ . Rotation through  $\frac{2\pi}{n}$  about this latter axis must evidently bring the net in the plane  $E_1 E_2 \dots E_n$  to self-coincidence since it brings the whole lattice to self-coincidence.

But a rotation-axis of a net passing through a point of the net can only be a 2-al, 4-al, or 6-al axis. Therefore :—

*A rotation-axis of a lattice which passes through one of its points is 2-al, 3-al, 4-al, or 6-al.*

§ 6. We shall always denote a primitive triplet of the translations represented by a lattice by  $2\tau_1, 2\tau_2, 2\tau_3$ .

In the lattice  $\Gamma_{tr}$  (whose symmetry is that of  $C_4$ ) the translations forming a primitive triplet are independent of one another in magnitude and direction.

If a lattice has a 2-al rotation-axis this axis must be parallel to a row of the lattice. Let  $C$  be a point on this row next to  $O$ , then  $OC$  ( $= 2\tau_z$ ) may be taken as one of a triplet of primitive translations of the lattice. Let  $OX$  ( $= 2\tau_1$ ) and  $OY$  ( $= 2\tau_2$ ) be the two other members of the triplet.

Let  $\epsilon_1, \epsilon_2$  be the two nets in the planes  $COX, COY$ ; then a primitive pair of translations of  $\epsilon_1$  is  $OX, OC$  and of  $\epsilon_2$  is  $OY, OC$ .

Now  $\epsilon_1, \epsilon_2$  must be brought to self-coincidence by a rotation through  $\pi$  about the axis  $OC$  lying in their plane (for the lattice as a whole is brought to self-coincidence by such a rotation); therefore these nets are either rectangular or rhombic.

lattice be brought to  $E_2, F_2, G_2, \dots$  by a rotation through  $\pi$  about  $l$ . Then  $E_1 E_2, F_1 F_2, G_1 G_2, \dots$  represent translations of the group perpendicular to  $l$ . Hence if  $E_1 F$  is parallel and equal to  $F_1 F_2$ ,  $E_1 G'$  to  $G_1 G_2$ , &c., the points  $E_1, E_2, F', G' \dots$  belong to the lattice and lie in a net perpendicular to  $l$ .

\* Of course a straight line passing through two points of a lattice passes through an infinite number of points of the lattice.

In all cases we see from Figs. 67, 68 that there are translations in the nets  $\epsilon_1, \epsilon_2$  perpendicular to  $OC$ . We shall denote these translations by  $2\tau_e, 2\tau_f$ . If  $\epsilon_1$  is rectangular  $OX = 2\tau_e$ , if  $\epsilon_1$  is rhombic  $OX = \tau_e + \tau_z$ ; and similarly in the case of  $\epsilon_2$ .

If  $\epsilon_1, \epsilon_2$  are both rectangular  $OC$  is perpendicular to both  $OX$  and  $OY$ , but  $OX$  is not in general perpendicular to  $OY$ . The lattice in this case may be considered as formed by placing points at the vertices of a series of right prisms with parallelogrammatic bases, which are similar and similarly situated and fill all space\*. We denote this kind of lattice by  $\Gamma_m$ ; for it

$$2\tau_1 = 2\tau_e; \quad 2\tau_2 = 2\tau_f, \quad 2\tau_3 = 2\tau_z \quad . \quad . \quad . \quad (1).$$

If  $\epsilon_1$  is rectangular and  $\epsilon_2$  rhombic,

$$2\tau_1 = 2\tau_e, \quad 2\tau_2 = \tau_f + \tau_z, \quad 2\tau_3 = 2\tau_z \quad . \quad . \quad (2),$$

and the case of  $\epsilon_1$  rhombic and  $\epsilon_2$  rectangular is of course geometrically indistinguishable from this.

If  $\epsilon_1$  and  $\epsilon_2$  are both rhombic,

$$2\tau_1 = \tau_e + \tau_z, \quad 2\tau_2 = \tau_f + \tau_z, \quad 2\tau_3 = 2\tau_z \quad . \quad . \quad (3).$$

We shall now show that cases (2) and (3) are geometrically indistinguishable.

In Fig. 72 let  $OA = 2\tau_e, OB = 2\tau_f$ ; bisect  $BC, CA, AB$  at  $A_1, B_1, C_1$ †; and let  $C_2$  be a point such that  $OC_2 = \tau_e - \tau_f$ .

Taking case (3) we have  $OX, OY$  coinciding with  $OB_1, OA_1$ ; so that  $OB_1, OA_1, OC$  represent a primitive triplet.

Now the tetrahedron  $OCA_1C_2$  is the tetrahedron  $OCA_1B_1$ , for they are on the same base and between the same parallels.

Therefore, since three non-coplanar edges of  $OCA_1B_1$  (i.e.  $OB_1, OA_1, OC$ ) represent a primitive triplet,  $OC_2, OA_1, OC$ , which are three non-coplanar edges of  $OCA_1C_2$ , also represent a primitive triplet.

Therefore  $OC_2, OC$  is a primitive pair in the plane  $OCC_2$ ; and the net  $OCC_2$  is rectangular. We are thus led back to case (2).

We denote this kind of lattice (2) or (3) by  $\Gamma'_m$ ; and proceed to find various primitive triplets belonging to it. We no longer confine ourselves to the case where  $OC$  is one of the triplet; but we still take  $2\tau_e$  and  $2\tau_f$  as the translations

\* That is to say, situated in the same way as the parallelepipeda of Fig. 66; each vertex (or angular point) is common to eight of the parallelepipeda.

† These symbols will not in future necessarily denote the same points as in Fig. 66.

perpendicular to  $OC$  and lying in planes through  $OC$  and two of the primitive triplet; and still denote  $OC$  by  $2\tau_z$ , and the primitive triplet by  $2\tau_1, 2\tau_2, 2\tau_3$ .

Since the triangle  $A_1OB = A_1OC$ ; therefore the tetrahedra  $C_2A_1OB, C_2A_1OC$  are equal, and hence  $OA_1, A_1B, OC_2$  represent a primitive triplet

$$2\tau_1 = \tau_f + \tau_z, 2\tau_2 = \tau_f - \tau_z, 2\tau_3 = 2\tau_e \dots (4).$$

The lattice may be considered as formed by placing points at the vertices of a series\* of prisms with rhombic bases and axes perpendicular to one of the diagonals of the base but not to the other ( $OA_1BC_2B_1C_1$  is half such a prism).

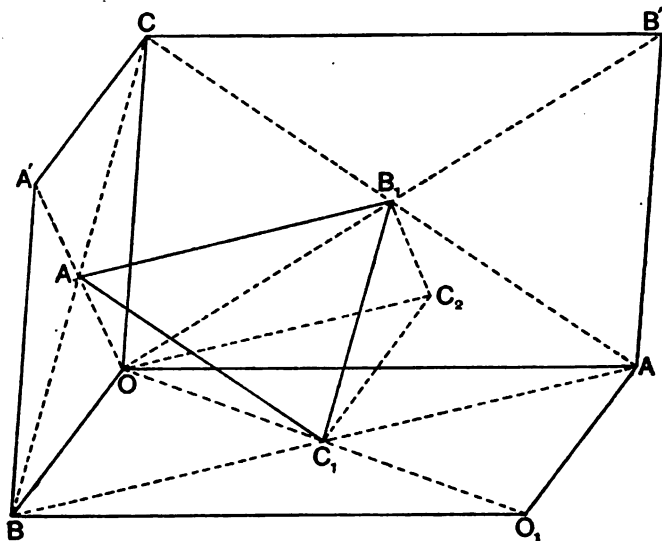


Fig. 72.

Again, Fig. 72 shows that the lattice may be formed by placing points at the vertices of a series of right prisms (of height  $OC$ ) with parallelograms (such as  $OC_2C_1B$ ) for their bases, and at the centres ( $A_1, B_1$ ) of two opposite side-faces of the prisms.

Since the tetrahedra  $OA_1B_1C_1, OA_1B_1C$  are equal, therefore  $OA_1, OB_1, OC_1$  represent a primitive triplet.

$$2\tau_1 = \tau_f + \tau_z, 2\tau_2 = \tau_z + \tau_e, 2\tau_3 = \tau_e + \tau_f \dots (5).$$

Hence the lattice may be formed by placing points at the

\* See footnote \*, p. 133.

vertices and at the centres ( $A_1, B_1, C_1$ , &c.) of *all* the faces of right prisms (of height  $OC$ ) with parallelograms (such as  $OAO_1E$ ) as their bases.

Again  $OC_1C_2 = AB_1C_2C_1 = C_1A_1OB = B_1A_1OC$ .

Therefore  $OC_1, OC_2, OB_1$  represent a primitive triplet.

$$2\tau_1 = 2\tau_o, 2\tau_2 = 2\tau_f, 2\tau_3 = \tau_o + \tau_f + \tau_s \quad . \quad . \quad (6).$$

Hence the lattice may be formed by placing points at the vertices and at the centres ( $B_1$ ) of right prisms (of height  $OC$ ) with parallelograms (such as  $OC_1AC_2$ ) as their bases.

Both the lattices described in this section have the symmetry of the group  $C_{2h}$ , for they both have a centre of symmetry at  $O$ , and a 2-al rotation-axis passing through  $O$ .

§ 7. Suppose now that the lattice has three mutually perpendicular rotation-axes  $OX, OY, OZ$  passing through  $O$ . Each of these axes passes through other points of the lattice; let  $2\tau_x, 2\tau_y, 2\tau_z$  be the primitive translations of the group represented by the lattice in the directions of these axes.

One type of lattice ( $\Gamma_o$ ) has these translations as a primitive triplet

$$2\tau_1 = 2\tau_x, 2\tau_2 = 2\tau_y, 2\tau_3 = 2\tau_z \quad . \quad . \quad . \quad (1).$$

This lattice may be formed by putting points at the vertices of a series of right prisms on rectangular bases; i.e. at the vertices of a series of right-angled parallelepipeda.

Now let any translation of any group represented by a lattice with  $OX, OY, OZ$  as three 2-al rotation-axes be  $\tau + \tau'$ , where  $\tau, \tau'$  are the components of the translation parallel and perpendicular to  $OZ$ . Then the group contains the translation  $\tau - \tau'$ , since  $OZ$  is a 2-al axis of the lattice. Therefore  $2\tau = (\tau + \tau') + (\tau - \tau')$  is a translation of the group, so that  $\tau$  is a multiple of  $\tau_z$ .

Similarly, the component of any translation parallel to  $OX$  is a multiple of  $\tau_x$ , and the component parallel to  $OY$  is a multiple of  $\tau_y$ . Therefore every translation of the lattice is of the form  $m\tau_x + n\tau_y + p\tau_z$ .

Combining any translation  $m\tau_x + n\tau_y + p\tau_z$  with multiples of  $2\tau_x, 2\tau_y, 2\tau_z$ , we see that every lattice with three orthogonal 2-al axes other than  $\Gamma_o$  must have one or more of the translations  $\tau_y + \tau_z, \tau_z + \tau_x, \tau_x + \tau_y, \tau_x + \tau_y + \tau_z$ ; and all possible lattices are found by combining one or more of these with  $2\tau_x, 2\tau_y, 2\tau_z$ .

Any net through a rotation-axis, and in particular each of the nets in the planes  $YOZ, ZOX, XOY$  must be either rhombic or rectangular. If one of them (say that in the plane

$XOY$ ) is rhombic,  $\tau_x + \tau_y, \tau_x - \tau_y$  are a primitive pair in the plane of this net. The lattice cannot in this case have the translation  $\tau_x + \tau_y + \tau_z$ , for then it would have

$$(\tau_x + \tau_y + \tau_z) - (\tau_x + \tau_y) = \tau_z$$

contrary to hypothesis. If it contains neither  $\tau_x + \tau_z$  nor  $\tau_y + \tau_z$ , the nets in the planes  $ZOX, ZOY$  are rectangular, and a primitive triplet of the lattice is

$$2\tau_1 = \tau_x + \tau_y, 2\tau_2 = \tau_x - \tau_y, 2\tau_3 = 2\tau_z \quad (2).$$

This lattice ( $\Gamma'_0$ ) may be formed by placing points at the vertices of a series of right prisms with rhombic bases (base  $OC_2AC_1$ , height  $OC$  in Fig. 72\*); or by placing points at the vertices and at the centres of the bases of right prisms with rectangular bases (base  $OA O_1B$ , height  $OC$ ).

If, however, a lattice for which the net in the plane  $XOY$  is rhombic contains the translations  $\tau_x + \tau_z$  and  $\tau_y + \tau_z$ †, then the nets in the planes  $XOZ, YOZ$  are also rhombic, and a primitive triplet of the lattice is

$$2\tau_1 = \tau_y + \tau_z, 2\tau_2 = \tau_x + \tau_z, 2\tau_3 = \tau_x + \tau_y \quad (3).$$

This lattice ( $\Gamma''_0$ ) may be formed by putting points at the vertices and at the centres of all the faces of a series of rectangular parallelepipeda (of which  $OA, OB, OC$  are concurrent edges).

If no one of the nets in the planes  $YOZ, ZOX, XOY$  is rhombic the lattice cannot contain  $\tau_y + \tau_z, \tau_x + \tau_z$ , or  $\tau_x + \tau_y$ . If it does not contain  $\tau_x + \tau_y + \tau_z$ , we have  $\Gamma_0$ ; if it does, we have a lattice ( $\Gamma'''_0$ ) such that

$$\tau_x + \tau_y + \tau_z \text{ and any two of } 2\tau_x, 2\tau_y, 2\tau_z \quad (4)$$

form a primitive triplet.

Other primitive triplets are

$$2\tau_1 = \tau_y + \tau_z - \tau_x, 2\tau_2 = \tau_x + \tau_z - \tau_y, 2\tau_3 = \tau_x + \tau_y - \tau_z \quad (5),$$

$$2\tau_1 = \tau_x + \tau_y + \tau_z, 2\tau_2 = \tau_x - \tau_y + \tau_z, 2\tau_3 = 2\tau_z \quad (6).$$

This lattice may be formed by placing points at the vertices and at the centres of the same series of rectangular parallelepipeda as were used to form  $\Gamma''_0$ .

The symmetry of all the lattices of this section is evidently that of  $D_{2h}$ , for they each have three 2-al axes and a centre of symmetry.

\* This figure applies to lattices of this type if  $OA$  is supposed to be perpendicular to  $OB$ , and if we take  $OA = 2\tau_x, OB = 2\tau_y, OC = 2\tau_z$ .

† If it contains one it contains the other, for

$$(\tau_x + \tau_z) + (\tau_y + \tau_z) = (\tau_x + \tau_y) + 2\tau_z.$$

Comparing the triplet (1) of this section with (1) of § 6, we see that  $\Gamma_o$  is a particular (or 'specialized') case of  $\Gamma_m$ ; similarly comparing (3) and (4) with (5) and (6) of § 6, we see that  $\Gamma_o''$  and  $\Gamma_o'''$  are specialized forms of  $\Gamma_m'$ ; and comparing (2) with (1) or (4) of § 6, we see that  $\Gamma_o'$  may be regarded as a specialized form of either  $\Gamma_m$  or  $\Gamma_m'$ .

§ 8. Suppose now that a lattice has a 4-al rotation-axis  $OZ$ . Then it has a square net perpendicular to  $OZ$  (p. 130); let  $2\tau_x, 2\tau_y$  be the smallest translations of the group represented by the lattice in the plane of this net. Then  $2\tau_x, 2\tau_y$  are a primitive pair, and are at right angles and equal to one another. Let the components of any other translation of the group parallel and perpendicular to  $OZ$  be  $\tau$  and  $\tau'$ . Then since  $OZ$  is a 4-al axis of the lattice there is a translation of the group whose components parallel and perpendicular to  $OZ$  are  $\tau$  and  $-\tau'$ . Therefore  $2\tau = (\tau + \tau') + (\tau - \tau')$  and  $2\tau' = (\tau + \tau') - (\tau - \tau')$  are translations of the group, so that  $\tau$  is a multiple of  $\tau_x$ , and  $\tau'$  is of the form  $m\tau_x + p\tau_y$ . Hence any translation of the group is of the form  $m\tau_x + n\tau_y + p\tau_z$ .

$m\tau_x + n\tau_y + p\tau_z - 2n\tau_y - 2p\tau_z = m\tau_x - n\tau_y - p\tau_z$  is also a translation of the group, so that the direction of  $\tau_x$  is a 2-al rotation-axis of the lattice; similarly the direction of  $\tau_y$  is a 2-al axis.

Hence lattices of this type have three mutually orthogonal rotation-axes and are therefore specialized forms of those of § 7.

$\Gamma_o$ , which is formed by placing points at the vertices of right prisms with rectangular bases, becomes, if  $OZ$  is a 4-al axis, a lattice formed by placing points at the vertices of right prisms with *square* bases (such as  $L'N'G'P'$   $LN'GP$ ,  $G'D'F'E'$   $GD'FE$  of Fig. 73 \*).

The figure shows that the lattice may also be formed by placing points at the vertices and at the centres of the bases of right prisms with square bases (such as  $N'D'E'P'$   $NDEP$ ,  $D'S'U'E'$   $DSUE$ ), proving that if  $OZ$  is a 4-al axis  $\Gamma_o$  and  $\Gamma_o'$  are identical.

A primitive triplet of this lattice  $\Gamma_t$  is

$$2\tau_1 = 2\tau_x, 2\tau_2 = 2\tau_y, 2\tau_3 = 2\tau_z \quad . \quad . \quad . \quad (1),$$

being the same as for  $\Gamma_o$  †.

$\Gamma_o''$  becomes a lattice formed by placing points at the

\* The points such as  $k, k$ , do not belong to the lattice in this case.

†  $\Gamma_o$  and  $\Gamma_t$  are distinguished by the fact that in the latter  $\tau_x$  and  $\tau_y$  are equal in magnitude.

vertices and at the centres of all the faces of right prisms with square bases (such as  $N'D'E'P'$   $NDEP$ ,  $D'S'U'E'$   $DSUE$ \*). The figure shows that this lattice may also be formed by placing points at the vertices and at the centres of right prisms with square bases (such as  $L'N'G'P'$   $LN'GP$ ,  $G'D'F'E'$   $GDFE$ ), proving that if  $OZ$  is a 4-al axis  $\Gamma_o''$  and  $\Gamma_o'''$  are identical.

A primitive triplet of this lattice  $\Gamma_t'$  is

$$2\tau_1 = \tau_y + \tau_z, 2\tau_2 = \tau_z + \tau_x, 2\tau_3 = \tau_x + \tau_y \quad . \quad . \quad (2),$$

being the same as for  $\Gamma_o'''$ .

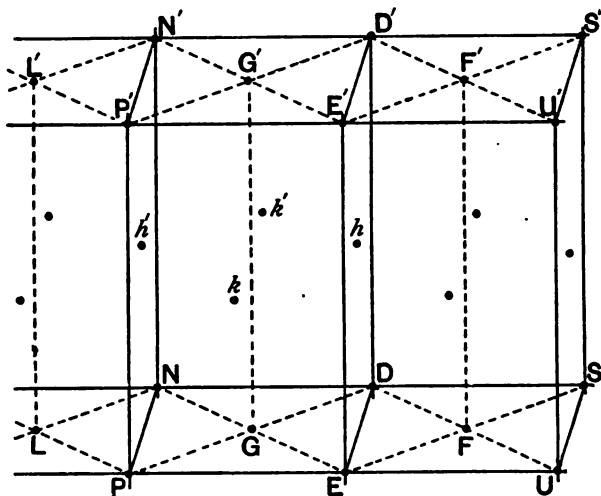


Fig. 78.

These lattices have the symmetry of the group  $D_{4h}$ , for they have a 4-al and a perpendicular 2-al rotation-axis passing through a centre of symmetry.

§ 9. Now suppose that the lattice has three mutually orthogonal 2-al axes as in § 7, but that  $\tau_x, \tau_y, \tau_z$  are all equal in magnitude.  $\Gamma_o$  may then be considered as formed by placing points at the vertices of a series of cubes. We denote the lattice in this case by  $\Gamma_r$ ; a primitive triplet is

$$2\tau_1 = 2\tau_x, 2\tau_2 = 2\tau_y, 2\tau_3 = 2\tau_z \quad . \quad . \quad . \quad (1).$$

$\Gamma_o'$ , which is formed by placing points at the vertices and at the centres of *one* pair of faces of rectangular parallelepipeda,

\* Now the points  $h, k$ , &c., belong to the lattice.

can have no symmetry when  $\tau_x, \tau_y, \tau_z$  are equal, which it does not have when  $\tau_x$  is equal to  $\tau_y$  but not to  $\tau_z$ .

$\Gamma_0''$  is now constructed by placing points at the vertices and at the centres of all the faces of a series of cubes, and is denoted by  $\Gamma_r'$ ; a primitive triplet is

$$2\tau_1 = \tau_y + \tau_z, 2\tau_2 = \tau_x + \tau_z, 2\tau_3 = \tau_x + \tau_y \quad (2).$$

$\Gamma_0'''$  is now constructed by placing points at the vertices and at the centres of a series of cubes, and is denoted by  $\Gamma_r''$ ; a primitive triplet is

$$2\tau_1 = \tau_y + \tau_z - \tau_x, 2\tau_2 = \tau_z + \tau_x - \tau_y, 2\tau_3 = \tau_x + \tau_y - \tau_z \quad (3).$$

$\Gamma_0''$  and  $\Gamma_0'''$  are not in this case identical, for if  $N'D'E'P' NDEP, D'S'U'E' DSUE$  are cubes (Fig. 73),  $L'N'G'P' LNGP, G'D'F'E' GDFE$  are not, and vice versa.

$\Gamma_r, \Gamma_r',$  and  $\Gamma_r''$  have evidently  $OX, OY, OZ$  as 4-al axes; their symmetry is that of  $O_h$ , for they each have three 4-al rotation-axes passing through a centre of symmetry.

§ 10. Suppose now that a lattice has a 3-al axis, and let  $OC = 2\tau_z$  be the primitive translation in its direction of the translation-group represented by the lattice.

First suppose that the component parallel to  $OC$  of every translation of the group is zero or a multiple of  $2\tau_z$ . Let  $2\tau', 2\tau''$  be two translations which with  $2\tau_z$  form a primitive triplet; and let their components parallel to  $OC$  be  $2m\tau_z, 2n\tau_z$  respectively. Then  $OE = 2\tau' - 2m\tau_z, OF = 2\tau'' - 2n\tau_z$  form with  $2\tau_z$  a primitive triplet (p. 122) and are perpendicular to  $OC$ . The net perpendicular to  $OC$  must be equilateral, since it is brought to self-coincidence by a rotation through  $\frac{2\pi}{3}$  about  $OC$ . Let  $OA = 2\tau_1, OB = 2\tau_2$  be a primitive pair of this net whose directions make an angle of  $\frac{2\pi}{3}$  with one

another. Then since the triangles  $OAB, OEF$  are equal, therefore the tetrahedra  $OABC, OEFC$  are equal, and hence  $2\tau_1, 2\tau_2, 2\tau_z$  are a primitive triplet of the lattice. The lattice ( $\Gamma_h$ ) may, in this case, be formed by placing points at the vertices of a series of right prisms (of height  $OC$ ) whose bases are rhombic with angles  $\frac{\pi}{3}$  and  $\frac{2\pi}{3}$ .

This arrangement is shown in Fig. 74; an inspection of the figure shows that  $OC$  is not only a 3-al but is also a 6-al rotation-axis, and that  $OA$  is a 2-al rotation-axis of the lattice.

Hence  $\Gamma_h$  has the symmetry of  $D_{6h}$ ; for it possesses a 6-al rotation-axis and a perpendicular 2-al axis passing through a centre of symmetry.

Now suppose that the translation-group represented by the lattice contains a translation whose projection on  $OC$  is not zero or a multiple of  $OC$ . Combining this translation with

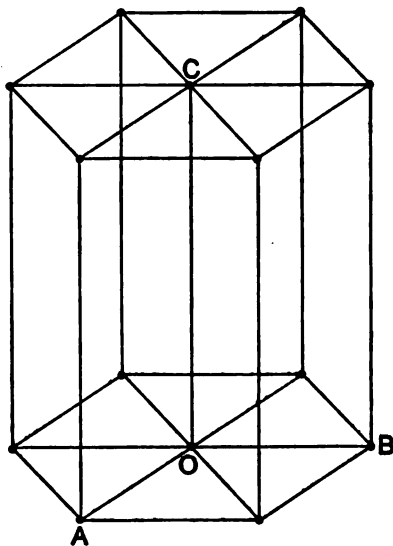


Fig. 74.

multiples of  $2\tau_z$  we see that there must be a translation whose projection on  $OC$  (or  $OC$  produced) is  $\geq \frac{1}{2}OC$ .

Let  $OA_1$  represent that translation whose projection on  $OC$  (or  $OC$  produced) is least\*. First suppose that  $OA_1$  makes an acute angle with  $OC$ , and let  $A_1$  be brought to  $A_2, A_3$  by rotation through  $\frac{2\pi}{3}, \frac{4\pi}{3}$  about  $OC$ .

Then  $OA_1 = 2\tau', OA_2 = 2\tau'', OA_3 = 2\tau'''$  are all translations of the group represented by the lattice, and therefore

$$2\tau' + 2\tau'' + 2\tau'''$$

is a translation of the group. It is evidently parallel to  $OC$ , and is therefore a multiple of  $2\tau_z$ ; for  $2\tau_z$  is the primitive translation in this direction. But the projections of  $OA_1, OA_2, OA_3$  on  $OC$  are all  $\geq \tau_z$ ; therefore

$$2\tau' + 2\tau'' + 2\tau''' \text{ is } \geq 3\tau_z$$

and is a multiple of  $2\tau_z$ , and hence  $2\tau' + 2\tau'' + 2\tau''' = 2\tau_z$ .

Similarly if  $OA_1$  makes an obtuse angle with  $OC$ ,

$$2\tau' + 2\tau'' + 2\tau''' = -2\tau_z;$$

and the following reasoning holds good if we substitute  $C'$  for  $C$ , where  $OC' = -2\tau_z$ .

New of all the translations whose projections on  $OC$  (or  $OC$  produced) are equal to that of  $OA_1$ , choose  $OA_1$  to be the one which makes with  $OC$  (or  $OC$  produced) an acute angle as small as possible.

\* Excluding translations whose component parallel to  $OC$  is zero.

Draw the parallelepipedon  $OC A_1 A_2 A_3 A_1' A_2' A_3'$  (Fig. 75); then, if  $Q$  be any point of the lattice, we see, by combining the translation  $OQ$  with multiples of  $2\tau'$ ,  $2\tau''$ ,  $2\tau'''$ , that there must be a point  $P$  of the lattice inside this parallelepipedon or upon its surface.

Now, if  $P$  does not coincide with  $O$  or  $C$ , it must lie in one of the planes  $A_1 A_2 A_3$ ,  $A_1' A_2' A_3'$ , for otherwise  $OP$ ,  $A_1 P$ , or  $CP$  would represent a translation whose projection on  $OC$  is less than that of  $OA_1$ . This being so  $P$  must coincide with  $A_1$ ,  $A_2$ ,  $A_3$ ,  $A_1'$ ,  $A_2'$ , or  $A_3'$ , for otherwise either  $OP$  or  $CP$  would make an angle with  $OC < COA_1$ .

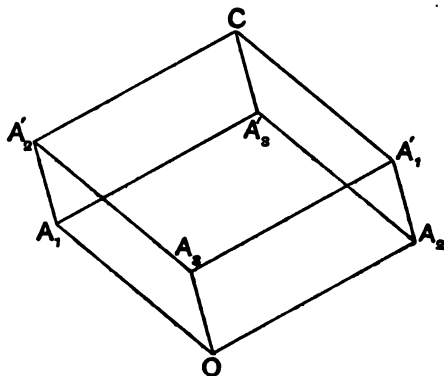


Fig. 75.

Hence  $2\tau'$ ,  $2\tau''$ ,  $2\tau'''$  are a primitive triplet of the lattice ( $\Gamma_{rh}$ ); which may be formed by placing points at the vertices of regular rhombohedra such as that of Fig. 75.

The plane  $COA_1$  is evidently a plane of symmetry for this lattice; it possesses a centre of symmetry lying on a 3-al rotation-axis, and a symmetry-plane through this axis, and has therefore the symmetry of  $D_{3d}^*$ .

§ 11. We have now proved that there are fourteen different kinds of lattice; it should be noticed that they all possess symmetry belonging to some *holohedral* class of crystals.

There is 1	{	whose symmetry is that	} triclinic system.
		of the holohedry of the	
There are 2	"	"	monoclinic system.
There are 4	"	"	orthorhombic system.
There are 2	"	"	tetragonal system.
There is 1	"	"	rhombohedral system.
There is 1	"	"	hexagonal system.
There are 3	"	"	regular system.

\* If  $\cos COA_1 = \sqrt{\frac{1}{3}}$ ,  $\sqrt{\frac{2}{3}}$ , or  $\frac{1}{2}$ ,  $\Gamma_{rh}$  becomes respectively  $\Gamma_r$ ,  $\Gamma_r'$ , or  $\Gamma_r''$ .

## CHAPTER XV

## THE BRAVAIS STRUCTURE-THEORY.

§ 1. The theory of crystal-structure put forward by Bravais in 1850 marks a very important stage in the historical development of the subject.

Bravais considers all molecules of a crystalline material to be alike and similarly orientated, and their centres of mass to be placed at the points of one of the fourteen possible lattices which we discussed in the previous chapter.

The individual molecules are assumed to themselves possess symmetry, and this symmetry is supposed to influence the symmetry of the medium as a whole.

Let  $O, A, B$  be any three points of a lattice, and let  $\omega, \alpha, \beta$  be the molecules whose centres of mass are at these three points.

Suppose  $l$  any axis of the first or second sort, then  $l$  evidently cannot be a symmetry-element of the medium as a whole unless it is an element of the lattice. Let  $l$  be a symmetry-element of the lattice, and let the corresponding operation leave  $O$  unmoved, bring  $A$  to  $B$ ,  $\omega$  to  $\omega'$ , and  $\alpha$  to  $\beta'$ .

Then since  $\omega, \alpha$  are similarly orientated, so are  $\omega', \beta'$ .

Now if  $l$  is a symmetry-element of  $\omega$  (as well as an element of the lattice),  $\omega'$  coincides with  $\omega$ , and therefore  $\beta'$  coincides with  $\beta$ . Hence a symmetry-element of the lattice and of one of the molecules is a symmetry-element of the structure as a whole; and hence is derived the statement that *the symmetry-elements of a crystal are those common both to the lattice and to one of the individual molecules.*

If we had supposed that the symmetry of the individual molecules had no effect on the symmetry of the crystal as a whole, or if we had supposed the molecules to be always spherical, we could only account for holohedral crystals; but by supposing that the molecules do affect the symmetry, and that the molecules can have the same symmetry as merohedral crystals, we can explain the existence of a crystal belonging to any one of the thirty-two classes, in spite of the fact that all lattices are of holohedral symmetry.

Theoretically we might have molecules of a low order of symmetry in a lattice of high order (thus we might have molecules of the symmetry of  $D_2$  in a lattice of the regular system; the crystal would then have the symmetry of  $D_2$  if the rotation-axes of any one of the molecules were parallel to the three 4-al axes of the lattice, the symmetry of  $C_2$  if only one of the axes of the molecule was parallel to a 4-al or 2-al axis of the lattice, and the symmetry of  $C_1$  if no axis of the molecule was parallel to a 2-al or 4-al axis of the lattice: or again we might have molecules of the symmetry of  $D_{2h}$  in the lattice  $\Gamma_{rh}$ , the crystal could then have at most the symmetry of  $C_{2h}$ ). Experience shows, however, that this does not often happen—crystals do not in general possess faces whose normals make the same angles with one another as in the case of crystals of higher symmetry—and there are also (according to Bravais) theoretical reasons why molecules should crystallize in a lattice whose symmetry is as like their own as possible (thus molecules, whose symmetry is that of  $C_3$ ,  $C_{3d}$ ,  $C_{3v}$ ,  $D_3$  or  $D_{3d}$ , usually crystallize in the lattice  $\Gamma_{rh}$  with their symmetry-elements parallel to the corresponding elements of the lattice; and the symmetry of the resulting crystal is the same as that of the individual molecules). An obvious objection to Bravais' theory is that the difficulty of explaining crystal-structure is only deferred (*reculée*), for we are left with the difficulty of explaining the symmetry of the individual molecules; Bravais' reply that 'the atomic theory furnishes us with an answer ready to hand, for it represents each molecule of a body as composed of a finite number of atoms of different sorts,' can hardly be considered satisfactory.

§ 2. It has been seen that in most systems more than one lattice is possible; Bravais tries to allot suitable lattices to certain crystals by means of the following considerations.

Firstly, cleavage surfaces are probably parallel to net-planes of the lattice in which points are most closely packed, and which consequently belong to sets whose 'interval' is comparatively large (p. 115).

Secondly, he gives reasons for assuming that the faces of most frequent occurrence on a crystal are also parallel to similar net-planes\*.

\* 'Car les mouvements internes qui agissent incessamment sur la superficie de la masse cristalline en voie de formation jouent, jusqu'à un certain point, le rôle des forces extérieures qui s'emploient à oliver un crystal; ils ont dû, par conséquent, respecter de préférence les séries moléculaires qui se trouvent disposées suivant des plans réticulaires dont les réseaux ont les tissus les plus serrés.'

Then he calculates the density of points in various net-planes of a lattice, and deduces the most probable cleavage and the forms of most frequent occurrence in crystals whose molecules are arranged at the points of such a lattice. By comparing his results with the cleavage and form of crystals found in nature he is able to assign to various crystals their appropriate lattices.

It is evident that the 'law of rational indices' is a direct conclusion from Bravais' theory that net-planes whose points are most closely packed are those parallel to the faces of most frequent occurrence, for it is exactly such faces whose indices can readily be expressed as comparatively small integers.

Again, we deduce from the theory that every symmetry-plane of a crystal is parallel to a possible face and that the line perpendicular to it is parallel to a possible edge; also that every symmetry-axis is parallel to a possible edge and that the plane perpendicular to it is parallel to a possible face\*. We deduce moreover that every symmetry-axis of a crystal is 2-al, 3-al, 4-al, or 6-al †. These deductions are in accordance with observation.

§ 3. Bravais' explanation of twinning is interesting. In those cases where two crystals belonging to a merohedral class are twinned in such a way that one is derived from the other by an operation (rotation about a 2-al axis, or reflexion in a plane) which is a symmetry-operation of the corresponding holohedral class ‡, he supposes that the two crystals have the *same* lattice, but that the molecules in the two members of the twin have different orientations; such twins are more or less inter-penetrant. In those cases where one member cannot be derived from the other by rotation about a 2-al axis, but only by reflexion in a plane the molecules of the two crystals must be enantiomorphous. If one member of a twin is derived from the other by rotation about a 2-al axis  $l$  which is not a symmetry-axis of the lattice, Bravais supposes that the two crystals possess similar but differently orientated lattices, one lattice being derived from the other by the same operation as in the case of the crystals themselves. The orientation of the molecules relatively to the lattice is the same in each crystal. The members of the twin do not inter-penetrate but are united along a net-plane perpendicular to  $l$ . This net belongs to both crystals for it is brought into self-coincidence

\* pp. 181, 182. The cases of a 3-al axis of the first sort and a 6-al axis of the second sort are no longer exceptional.

† p. 182.

‡ This is a case often met with; it is called 'supplementary twinning.'

by a rotation through  $\pi$  about  $L$ . Bravais admits that there are some cases of twinning which are not readily explained on his theory.

§ 4. Bravais' structure-theory is interesting historically, for it has had very considerable influence on subsequent developments of the subject. It is not at the present day considered quite satisfactory, since, in addition to objections already mentioned, it does not include all possible or probable arrangements of molecules or atoms which may be suggested as representing the structure of various crystals. In particular, as Sohncke pointed out, an arrangement of molecules similar to that of the vertices of the cells in a honeycomb is not provided for in Bravais' theory; but such an arrangement is simple and might well represent the structure of certain crystals. Before we can discuss other theories we must investigate the number and properties of infinite groups of movements; this investigation will occupy chapters xvi to xxiii.

## CHAPTER XVI

## PROPERTIES OF GEOMETRICAL OPERATIONS.

§ 1. We proved in chapter iii that any operation which can bring a figure into coincidence with a congruent or enantiomorphous figure is equivalent to a translation, rotation, rotatory-reflexion, or to a combination of a translation and a rotation. Since the 'product' of a series of operations of this kind must also be an operation bringing the figure into coincidence with a congruent or enantiomorphous figure, the product is itself a rotatory-reflexion, a translation, a rotation, or a combination of the last two. We shall proceed to find the result of combining various operations in some useful cases.

§ 2. *A rotation  $A(a)$  through an angle  $a$  about an axis  $a$ , and a translation  $T$  parallel to  $a$  are permutable operations.*

For let  $P$  be any point, and let  $P$  be brought to  $P_1$  by  $A(a)$ ; let  $P, P_1$  be brought to  $P_2, P_3$  by  $T$ . Then evidently  $P$  is brought to  $P_1$  and then to  $P_3$  by  $T$  followed by  $A(a)$ , and to  $P_2$  and then to  $P_3$  by  $A(a)$  followed by  $T$ ; but  $P$  is any point, and  $\therefore T.A(a) = A(a).T$ .

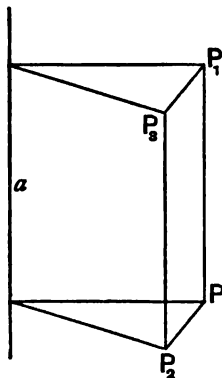


Fig. 76.

The combination of a rotation about an axis and a translation parallel to it is called a *screw* about that axis; and if such a combination brings a figure  $U$  to self-coincidence the axis is called a *screw-axis of symmetry* for  $U$ , or more briefly a *screw-axis* of  $U$ . If the angle of rotation about the axis ( $a$ ) is  $a$ , and the length of the translation is  $t$ , the screw about  $a$  is usually represented by the symbol  $A_{a,t}$ .

This screw will be called 'a screw of angle  $a$  and translation  $t$ '; or, if  $na = 2\pi$ , 'an  $n$ -al screw of translation  $t$ .'

A screw is said to be right-handed or left-handed according as, when we look along the axis in the direction of the translation, the rotation appears to be in the positive (clock-wise) or negative direction.

§ 3. *Any translation  $T$ , followed by a rotation  $A(\omega)$  about an axis  $a$ , perpendicular to the translation, is equivalent to a rotation through the same angle about a parallel axis.*

Let  $A_1$  be any point on  $a$ ; let  $C$  be a point such that  $CA_1$  represents the translation. Take a point  $B_1$  such that the angle  $B_1CA_1 = CA_1B_1 = \frac{\pi - \omega}{2}$ , and draw  $B_1D$  parallel and equal to  $CA_1$ . Then  $CA_1DB_1$  is a parallelogram and the angle  $A_1B_1C = B_1A_1D = \omega$ .

Now  $T$  brings the points  $C, B_1$  to  $A_1, D$  respectively, and  $A(\omega)$  brings  $A_1, D$  to  $A_1, B_1$  respectively. Therefore  $T.A(\omega)$  brings the points  $C, B_1$  to  $A_1, B_1$  respectively.

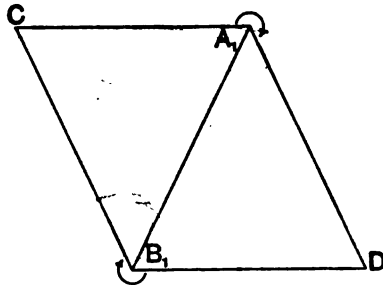


Fig. 77.

Now a rotation  $B(\omega)$  about an axis  $b$  through  $B_1$  parallel to  $a$  does the same; therefore  $T.A(\omega)$  and  $B(\omega)$  bring three non-collinear points (viz.  $C$ , and any two points on  $a$ ) into the same positions, and therefore  $T.A(\omega)$  brings any figure into the same position as  $B(\omega)$  does; hence  $T.A(\omega) = B(\omega)$ .

Again, since  $T.A(\omega) = B(\omega)$ , therefore

$B(-\omega).T.A(\omega) = 1$  (for  $\{B(\omega)\}^{-1} = B(-\omega)$ ), and  $B(-\omega).T = A(-\omega)$ , or  $B(\phi).T = A(\phi)$  (putting  $\phi$  for  $-\omega$ ).

Hence, *a rotation about any axis, followed by a translation perpendicular to that axis, is equivalent to a rotation through the same angle about a parallel axis.*

Again, since  $T.A(\omega) = B(\omega)$ , therefore  $B(\omega).A(-\omega) = T$ .

Hence, *a rotation about any axis, followed by a rotation through the same angle in the opposite direction about a parallel axis, is equivalent to a translation perpendicular to the axes.*

This is also evident from Euler's construction (p. 24) for

a rotation through an infinitesimal angle about an infinitely distant axis is equivalent to a translation perpendicular to that axis.

From Fig. 77 we see that  $CA_1 = 2A_1B_1 \sin \frac{\omega}{2}$ ; we deduce from this that if  $\omega = \pi$ ,  $CA_1 = 2A_1B_1$  and the angle  $CA_1B_1 = 0$ ; that is  $B_1$  bisects the line  $CA_1$ . Hence a translation  $A_{-1}A_1$ , followed by a rotation through  $\pi$  about an axis through  $A_1$  (see Fig. 64), is equivalent to a rotation through  $\pi$  about a parallel axis through  $O$ ; a rotation through  $\pi$  about an axis through  $A_{-1}$ , followed by a similar rotation about a parallel axis through  $O$ , is equivalent to the translation  $A_{-1}A_1$ , &c.

If  $\omega = \frac{2\pi}{3}$ ,  $CA_1 = \sqrt{3} \cdot A_1B_1$  and  $CA_1B_1 = \frac{\pi}{6}$ ; that is,  $B_1$  is at the centre of an equilateral triangle  $CA_1E$  whose side is  $CA_1$  (Fig. 78). We have

$$CA_1 \cdot A\left(\frac{2\pi}{3}\right) = B\left(\frac{2\pi}{3}\right),$$

$$B\left(\frac{2\pi}{3}\right) \cdot EA_1 = A\left(\frac{2\pi}{3}\right),$$

$$A\left(\frac{2\pi}{3}\right) \cdot B\left(-\frac{2\pi}{3}\right) = A_1C,$$

$$A\left(-\frac{2\pi}{3}\right) \cdot B\left(\frac{2\pi}{3}\right) = A_1E, \text{ \&c., \&c.}$$

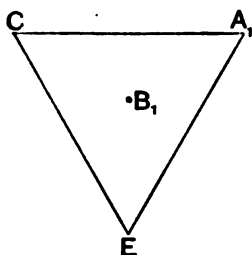


Fig. 78.

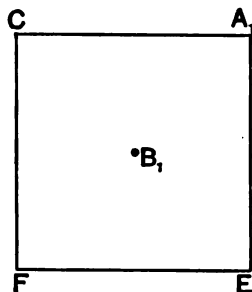


Fig. 79.

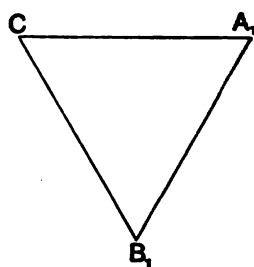


Fig. 80.

If  $\omega = \frac{\pi}{2}$ ,  $CA_1 = \sqrt{2} \cdot A_1B_1$  and  $CA_1B_1 = \frac{\pi}{4}$ ; that is,  $B_1$  is at the centre of a square  $CA_1EF$  whose side is  $CA_1$  (Fig. 79).

We have

$$CA_1 \cdot A\left(\frac{\pi}{2}\right) = B\left(\frac{\pi}{2}\right),$$

$$B\left(\frac{\pi}{2}\right) \cdot EA_1 = A\left(\frac{\pi}{2}\right),$$

$$A\left(\frac{\pi}{2}\right) \cdot B\left(\frac{-\pi}{2}\right) = A_1C,$$

$$A\left(\frac{-\pi}{2}\right) \cdot B\left(\frac{\pi}{2}\right) = A_1E, \text{ \&c., \&c.}$$

If  $\omega = \frac{\pi}{3}$ ,  $CA_1 = A_1B_1$  and  $CA_1B_1 = \frac{\pi}{3}$ ; that is,  $B_1$  is at the third vertex of an equilateral triangle whose side is  $CA_1$  (Fig. 80). We have

$$CA_1 \cdot A\left(\frac{\pi}{3}\right) = B\left(\frac{\pi}{3}\right),$$

$$B\left(\frac{\pi}{3}\right) \cdot B_1C = A\left(\frac{\pi}{3}\right),$$

$$A\left(\frac{\pi}{3}\right) \cdot B\left(\frac{-\pi}{3}\right) = A_1C,$$

$$A\left(\frac{-\pi}{3}\right) \cdot B\left(\frac{\pi}{3}\right) = CB_1, \text{ \&c., \&c.}$$

§ 4. Any translation  $T$ , followed by a rotation  $A(\omega)$  about an axis  $a$ , is equivalent to a screw  $B_{\omega, t}$  about some axis parallel to  $a$ , where  $t$  is the component of  $T$  in the direction of  $a$ .

For let  $T$  be resolved into components  $t, t'$  parallel and perpendicular to  $a$ . Then  $T \cdot A(\omega)$  is equivalent to a translation  $t$  parallel to  $a$ , followed by a translation  $t'$  perpendicular to  $a$ , and a rotation through  $\omega$  about  $a$ ; i. e. to a translation  $t$  parallel to  $a$ , followed by a rotation through  $\omega$  about some axis  $b$  parallel to  $a$  (by § 3). Therefore  $T \cdot A(\omega) = B_{\omega, t}$ . Similarly we can prove  $A(\omega) \cdot T = B'_{\omega, t}$ .

§ 5. We can now state the result quoted at the beginning of this chapter in the form:—

*'Every operation which brings a figure into coincidence with a congruent figure is equivalent to a screw, and every operation which brings a figure into coincidence with an enantiomorphic figure is equivalent to a rotatory-reflexion.'*

A screw is said to be an operation of the first sort, and a rotatory-reflexion an operation of the second sort.

The product of any number of screws and rotatory-reflexions is equivalent to a screw or rotatory-reflexion, according as the number of rotatory-reflexions in the product is even or odd\*.

The following particular cases of the screw  $A_{\omega, t}$  are interesting:—

- (1)  $\omega = 0$ ; the screw becomes a translation.
- (2)  $t = 0$ ; the screw becomes a rotation.

The following particular cases of a rotatory-reflexion are interesting:—

(1) Angle of rotation  $= 0$ ; the rotatory-reflexion becomes a simple reflexion.

(2) Angle of rotation  $= \pi$ ; the rotatory-reflexion becomes an inversion about the intersection of the axis and plane of the rotatory-reflexion.

(3) Angle of rotation infinitesimal and axis at an infinite distance; the rotatory-reflexion becomes a reflexion in a plane combined with a translation parallel to that plane†.

The plane is called, in this case, a *gliding-plane* and the operation a *gliding-reflexion*. A gliding-reflexion whose gliding-plane is  $\sigma$ , and whose translation (parallel to  $\sigma$ ) is represented by  $t$  is usually denoted by the symbol  $S(t)$ .

§ 6. We now proceed to find the result of combining various operations in some useful cases.

First take the case of a screw  $A_{\omega, t}$  about an axis  $a$ , and any translation  $T$ ; let the components of  $T$  parallel and perpendicular to  $a$  be  $t'$  and  $t''$ . Then  $A_{\omega, t}.T$  is equivalent to a rotation through  $\omega$  about  $a$ , followed by a translation  $t$  parallel to  $a$ , another translation  $t'$  parallel to  $a$ , and a translation  $t''$  perpendicular to  $a$ ; that is, to a rotation through  $\omega$  about  $a$ , followed by a translation  $t''$  perpendicular to  $a$ , and a translation  $t+t'$  parallel to  $a$  (for all translations are permutable operations‡); that is, to a rotation through  $\omega$  about an axis  $b$  parallel to  $a$ , followed by a translation  $t+t'$  parallel to  $a$  or  $b$ ; that is, to a screw  $B_{\omega, t+t'}$  about  $b$ .

Similarly we may prove  $T.A_{\omega, t} = B'_{\omega, t+t'}$  about some other axis  $b'$  parallel to  $a$ .

\* p. 80.

† Since rotation about an axis and reflexion in a perpendicular plane are permutable (p. 81), proceeding to the limit when the axis is at infinity and the angle of rotation infinitesimal, we see that a reflexion in a plane and a translation parallel to that plane are permutable operations. This may easily be proved independently.

‡ See p. 118.

Now we shall take the case of two screws  $A_{a,t_1}, B_{\beta,t_2}$  about two parallel axes  $a, b$ .

Then  $A_{a,t_1} \cdot B_{\beta,t_2}$  = a translation  $t_1$  parallel to  $a$ , followed by rotations  $A(a), B(\beta)^*$ , and a translation  $t_2$  parallel to  $b$  (or  $a$ ). But  $A(a) \cdot B(\beta)$  is equivalent to a rotation  $C(a+\beta)$  about some axis  $c$  parallel to  $a$  and  $b$  (p. 24); therefore  $A_{a,t_1} \cdot B_{\beta,t_2}$  is equivalent to the screw  $C_{a+\beta,t_1+t_2}$ , for  $C(a+\beta)$  is permutable with translations parallel to  $c$  (p. 146).

Putting  $a+\beta = 0$ , we see that the product of screws  $A_{a,t_1}, B_{-a,t_2}$  about two parallel axes  $a$  and  $b = a$  translation whose component parallel to  $a$  or  $b$  is  $t_1+t_2$ .

§ 7. Let  $C_{\gamma,t}$  and  $C'_{\gamma,0}$  be a screw and a rotation about two parallel axes. Then since  $C_{\gamma,t} \cdot C'_{-\gamma,0}$  = some translation  $T$ ,

$$\therefore C_{\gamma,t} = T \cdot (C'_{-\gamma,0})^{-1} = T \cdot C'_{\gamma,0}.$$

Again, since  $C'_{-\gamma,0} \cdot C_{\gamma,t}$  = some translation  $T'$ ,

$$\therefore C_{\gamma,t} = C'_{\gamma,0} \cdot T' = T \cdot C'_{\gamma,0}.$$

Conversely if  $C'_{\gamma,0}$  is any rotation and  $T$  any translation, a translation  $T'$  can be found such that  $T \cdot C'_{\gamma,0} = C'_{\gamma,0} \cdot T'$ .

§ 8. We proceed now to the more general case of any two screws  $A_{a,t_1}, B_{\beta,t_2}$  about any two axes  $a$  and  $b$ .

Take any point  $P$  and draw lines  $a', b'$  through  $P$  parallel to  $a$  and  $b$ . Let  $C'_{\gamma,0}$  be the rotation about an axis through  $P$  which is equivalent to  $A'(a) \cdot B'(\beta)$ .

Then by § 7  $A_{a,t_1} \cdot B_{\beta,t_2} = T_1 \cdot A'_{a,0} \cdot B'_{\beta,0} \cdot T_2 = T_1 \cdot C'_{\gamma,0} \cdot T_2 = C'_{\gamma,0} \cdot T' \cdot T_2 = C'_{\gamma,0} \cdot T$  (similarly)  $T \cdot C'_{\gamma,0}$ , where  $T_1, T_2, T', T$  are certain translations.

Repetition of this process shows that the product of any screws  $A_{a,t_1} \cdot B_{\beta,t_2} \cdot C_{\gamma,t_3} \cdot D_{\delta,t_4} \dots$  about axes  $a, b, c, d, \dots$  =  $M'_{\mu,0}$  combined with some translation, where  $M'_{\mu,0}$  is the result of combining rotations through  $a, \beta, \gamma, \delta, \dots$  about axes through any point respectively parallel to  $a, b, c, d, \dots$ .

The practical method of finding the resultant of any number of screws is as follows: take a point  $Q$  which is brought by  $A_{a,t_1} \cdot B_{\beta,t_2} \cdot C_{\gamma,t_3} \dots$  into an easily ascertained position  $P$ . Find, by Euler's construction, the position of the axis ( $m'$ ) and the angle of rotation ( $\mu$ ) of the resultant of rotations through  $a, \beta, \gamma, \dots$  about axes through  $P$  parallel to  $a, b, c, \dots$  respectively. Then  $A_{a,t_1} \cdot B_{\beta,t_2} \cdot C_{\gamma,t_3} \dots$  is equivalent to some

\* These can also be written  $A_{a,0}, B_{\beta,0}$ .

translation  $T$  followed by the rotation  $M'_{\mu,0}$  about  $m'$ , and therefore  $T \cdot M'_{\mu,0}$  brings  $Q$  to  $P$ .

Hence (since  $P$  lies on  $m'$ )  $T$  is the translation  $QP$ . We have now found completely the operations  $T$  and  $M'_{\mu,0}$ . Let  $t$  and  $t'$  be the components of  $T$  parallel and perpendicular to  $m'$ . Find by the construction of p. 147 the position of the axis  $m$  such that  $M_{\mu,0} =$  the translation  $t'$  followed by the rotation  $M'_{\mu,0}$ .

Then  $A_{\alpha,t_1} \cdot B_{\beta,t_2} \cdot C_{\gamma,t_3} \dots =$  the screw  $M_{\mu,t}$  about the axis  $m$ .

A useful example is the following:—

'Find the resultant of a screw  $A_{\pi,\pi_a}$  about an axis  $a$ , followed by a screw  $B_{\pi,\pi_b}$  about an axis  $b$ , if  $a$  is at a distance  $t_c$  from  $b$  and makes an angle  $\theta$  with it.'

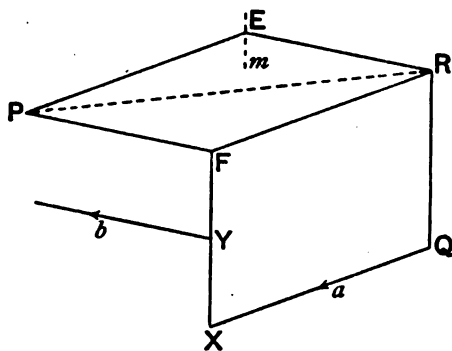


Fig. 81.

Let the line perpendicular to  $a$  and  $b$  meet them in  $X$  and  $Y$ ; then  $XY = t_c$ . Produce  $XY$  to  $F$  making  $YF = XY$ , and draw  $FR$ ,  $FP$  parallel to  $a$  and  $b$  making

$$RF = 2t_a, \quad FP = 2t_b.$$

Draw  $RQ$  parallel to  $YX$  meeting  $a$  in  $Q$ ; then  $QX = 2t_a$ .

Make the angles  $RPE$ ,  $ERP$  in the plane  $FPR$  each  $= \frac{\pi}{2} - \theta$ . (Fig. 81).

Then  $A_{\pi,\pi_a}$  brings  $Q$  to  $X$ , and  $B_{\pi,\pi_b}$  brings  $X$  to  $P$ .

$$\therefore A_{\pi,\pi_a} \cdot B_{\pi,\pi_b} \text{ brings } Q \text{ to } P.$$

Now rotations through  $\pi$ , first about an axis through  $P$  parallel to  $a$ , and then about an axis through  $P$  parallel to  $b$ , are equivalent to a rotation through  $2\theta$  about an axis through  $P$  parallel to  $XY$  (p. 25). Hence  $A_{\pi,\pi_a} \cdot B_{\pi,\pi_b} =$  the translation  $QP$  (that is, a translation  $QR$  and a translation  $RP$ ), followed by a rotation through  $2\theta$  about an axis through  $P$  parallel to  $XY =$  the translation  $QR (= 2t_c)$  followed by a rotation  $M_{2\theta,0}$  about an axis  $m$  through  $E$  parallel to  $XY$  (p. 147)  $= M_{2\theta,\pi_c}$ .

The distance of  $m$  from  $a$  = the distance of  $E$  from  $FR$

$$\begin{aligned}
 &= ER \cdot \sin\left(\frac{\pi}{2} - \theta + PRF\right) = ER \cdot \cos(\theta - PRF) \\
 &= \frac{PR \cdot \cos(\theta - PRF)}{2 \sin \theta} = \frac{PR \cdot \cos RPF}{2 \sin \theta} \\
 &= \frac{PF - FR \cdot \cos RFP}{2 \sin \theta} = \frac{t_b + t_a \cos \theta}{\sin \theta}.
 \end{aligned}$$

Similarly the distance of  $m$  from  $b$  =  $\frac{t_a + t_b \cos \theta}{\sin \theta}$ .

The following special cases are of importance:—

$$(1) \quad \theta = \frac{\pi}{2}.$$

'The product of three screws of angle  $\pi$  whose axes are three perpendicular sides (of lengths  $t_a, t_b, t_c$ ) of a rectangular

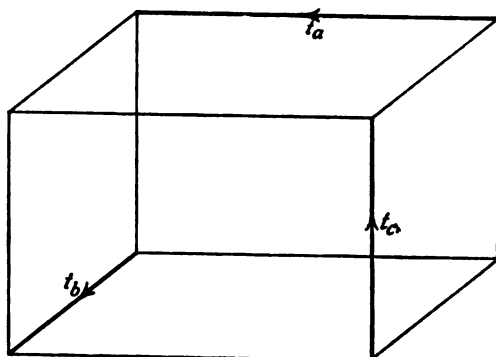


Fig. 82.

parallelepiped which do not cut, and whose translations are twice the respective sides taken in order\*, is unity.'

$$(2) \quad t_a = t_b = 0.$$

'Two rotations through  $\pi$  about axes making an angle  $\theta$  with one another, and at a distance  $t_c$  apart, are equivalent to a screw of angle  $2\theta$  and translation  $2t_c$  about the line meeting and perpendicular to both axes.'

The case of  $\theta = \frac{\pi}{2}$  is of especial importance.

$$(3) \quad t_c = 0, \theta = \frac{\pi}{2}.$$

\* That is, taken in directions such that  $t_a + t_b + t_c$  = a translation represented by that diagonal of the parallelepiped which meets none of the three axes. Figure 82 makes this clear.

'Two screws of angle  $\pi$  and translations  $2t_a$  and  $2t_b$  about perpendicular intersecting axes  $a$  and  $b$  are equivalent to a rotation through  $\pi$  about an axis perpendicular to  $a$  and  $b$ , and at distances  $t_b, t_a$  from  $a, b$  respectively.'

(4) Case (2) may be put in the following form:—

'A rotation through  $\pi$  about an axis  $a$ , followed by a screw of angle  $2\theta$  and translation  $2t_c$  about a line  $c$  perpendicular to  $a$  and meeting it, is equivalent to a rotation through  $\pi$  about an axis perpendicular to and meeting  $c$ , and making an angle  $\theta$  with and at a distance  $t_c$  from  $a$ .'

§ 9. We shall now consider operations of the second sort.

Suppose we have reflexions  $S, S_1$  in two parallel planes  $\sigma, \sigma_1$ ; then  $S.S_1$  is equivalent to a translation perpendicular to  $\sigma$  and  $\sigma_1$  through a distance equal to twice that between  $\sigma$  and  $\sigma_1$ .

For let  $P$  be any point,  $Q$  its reflexion in  $\sigma$ , and  $R$  the reflexion of  $Q$  in  $\sigma_1$ . Let  $RPQ$  meet  $\sigma, \sigma_1$  in  $H$  and  $H_1$  (Fig. 83).

Then

$$\begin{aligned} PR &= PH_1 + H_1R = PH_1 + QH_1 \\ &= 2PH_1 + QP = 2PH_1 + 2HP \\ &= 2HH_1; \end{aligned}$$

so that any point  $P$  is translated by  $S.S_1$  through a distance  $2HH_1$  perpendicular to  $\sigma$  and  $\sigma_1$ .

It will be seen that the above is a limiting case of the theorem of p. 26.

§ 10. There is now no difficulty in combining any number of screws and rotatory-reflexions.

First consider a rotatory-reflexion of angle  $\alpha$  whose axis is  $a$  and whose plane is  $\sigma$ . Take any point  $P$  and through it draw  $a'$  parallel to  $a$ , and  $\sigma'$  parallel to  $\sigma$ . Let  $S, S'$  denote reflexions in  $\sigma$  and  $\sigma'$ , and let  $A(a), A'(a)$  denote rotations through  $\alpha$  about  $a$  and  $a'$ .

Then  $S.S'$  is some translation  $T_1$  (§ 9).

$\therefore S = T_1.S'$ , since  $S'^2 = 1$ .

Again  $A(a) = A'(a).T_2$ , where  $T_2$  is some translation (p. 147). Therefore the rotatory-reflexion about  $a$  and  $\sigma$ , namely,  $A(a).S = A'(a).T_2.T_1.S' = A'(a).T'.S'$ , where  $T'$  is some translation  $= T'.A'(a).S'$ , where  $T'$  is some translation (see

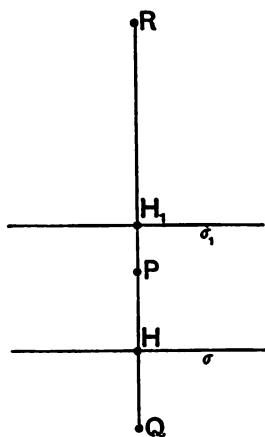


Fig. 88.

§ 7, p. 151) = a similar rotatory-reflexion about  $a'$  and  $\sigma'$  multiplied by some translation or other.

Similarly  $A(a).S = A'(a).S'.T$  where  $T$  is some other translation.

Now consider the resultant of a screw  $B_{\beta,t}$  and the rotatory-reflexion  $A(a).S$ .

Let  $B'(\beta)$  be a rotation through  $\beta$  about a line through the point  $P$  parallel to the axis of  $B_{\beta,t}$ . Then  $B_{\beta,t} = B'(\beta).T''$ , where  $T''$  is some translation (p. 151).

Therefore

$$B_{\beta,t}.A(a).S = B'(\beta).T''.T.A'(a).S' = T'.B'(\beta).A'(a).S^*$$

where  $T'$  is some translation or other.

This can be immediately extended, and we obtain :—

*The resultant  $R$  of any number of screws and rotatory-reflexions is found by multiplying by some translation or other the resultant  $R'$  of rotations and rotatory-reflexions of the same angles about parallel axes and planes through any given point.*

In this theorem rotations are included as particular cases of screws, and reflexions and inversions as particular cases of rotatory-reflexions.

We may put the theorem in the form  $R = T.R' = R'.T$ .

We have shown in Part I how to obtain  $R'$ ; the *practical* method of finding the resultant of any number of operations is exactly similar to that given on p. 151 for finding the resultant of any number of screws.

§ 11. There are two special cases of the above theorem whose simplicity and importance demand further attention. Independent proofs similar to that given for reflexions in two parallel planes (§ 9) can be at once obtained. They are:—

(1) The resultant of two inversions  $I, I_1$  about points  $H, H_1$  is a translation  $T$  represented by  $2HH_1$ .

Since  $I^2 = I_1^2 = 1$  and  $I.I_1 = T$ ; therefore  $I = T.I_1$ ,  $I_1 = I.T$ .

(2) The resultant of a reflexion  $S$  in a plane  $\sigma$  and an inversion  $I_1$  about a point  $H_1$  is a screw  $E$  about the line  $HH_1$  of angle  $\pi$  and translation  $2HH_1$ , where  $H$  is the foot of the perpendicular from  $H_1$  on  $\sigma$ .

Since  $S^2 = I_1^2 = 1$  and  $S.I_1 = E$ ; therefore  $I_1 = S.E$ ,  $S = E.I_1$ .

\* See § 7 on p. 151.

§ 12. If  $A, B$  are any two operations, then  $A^{-1}.B.A$  is called *the result of transforming  $B$  by  $A$* .

Let  $L$  denote the axis\* of the operation  $B$ , and let  $L'$  be the position into which  $L$  is brought by the operation  $A$ .

Now let any point  $c'$  be brought to  $c$  by  $A^{-1}$ ,  
and let  $c$  be brought to  $d$  by  $B$ ,  
and let  $d$  be brought to  $d'$  by  $A$ .

Then  $c'$  is brought to  $d'$  by  $A^{-1}.B.A$ .

Now since the point  $c$  is brought to  $c'$  by  $A$ ,  
and the point  $d$  is brought to  $d'$  by  $A$ ,  
and the axis  $L^*$  is brought to  $L'$  by  $A$ ;

therefore the figure  $L, c, d$  is congruent or enantiomorphous to  $L', c', d'$  according as  $A$  is of the first or second sort.

Now  $B$  acting about  $L$  brings  $c$  to  $d$ ; and therefore an operation acting about  $L'$  precisely similar to  $B$  (but whose angle of rotation is equal or equal and opposite to that of  $B$ , according as  $A$  is of the first or second sort) must bring  $c'$  to  $d'$ .

But  $A^{-1}.B.A$  also brings  $c'$  to  $d'$ , and the above is true for all positions of the point  $c'$ . Hence  $A^{-1}.B.A$  must be identical with the operation similar to  $B$  (but whose angle of rotation is equal or equal and opposite to that of  $B$ , according as  $A$  is of the first or second sort) acting about the position into which the axis\* of the operation  $B$  is brought by the operation  $A$ .

For example, if  $B$  is a right-handed screw,  $A^{-1}.B.A$  is a screw of the same translation and angle, right-handed or left-handed, according as  $A$  is an operation of the first or second sort.

If an operation  $A$  brings a figure  $U$  to coincidence with  $U'$ ,  $U'$  is said to be the result of transforming  $U$  by  $A$ .

\* Or the system axis + plane if  $B$  is of the second sort.

## CHAPTER XVII

## INFINITE GROUPS OF MOVEMENTS.

§ 1. *Every group of movements which brings a collection of molecules forming a crystal into self-coincidence has as a subgroup one of the fourteen translation-groups considered in chapter xiv.*

This follows from the definition of crystalline media given on p. 113. For if  $O$ ,  $A$  be two points of the medium not very close together, then there must be a point  $A'$  such that  $AA'$  is small compared with  $OA$ , and such that the medium considered as a geometrical figure bears exactly the same relation to  $A'$  that it does to  $O$ ; that is, such that the translation  $OA'$  brings the crystal to self-coincidence. Similarly, taking two points  $B$ ,  $C$  not too close to  $O$ , and such that  $OA$ ,  $OB$ ,  $OC$  are not nearly coplanar, we have translations belonging to the group of movements which bring the medium to self-coincidence, represented by lines  $OB'$ ,  $OC'$  differing little from  $OB$ ,  $OC$  in magnitude and position. We have then three translations  $OA'$ ,  $OB'$ ,  $OC'$  belonging to the group which are not coplanar, and the group can contain no infinitesimal translation; therefore the group must include a subgroup of translations of one of the fourteen kinds discussed on pp. 132 to 141.

§ 2. If a line  $l$  of given length and direction is brought to coincide with  $l'$  by any operation  $H$ , then the translation represented by  $l$  is said to be brought by  $H$  into coincidence with the translation represented by  $l'$ . If  $H$  brings every member of any system of translations into coincidence with itself or some other member of the system,  $H$  is said to bring the system of translations to self-coincidence.

A translation evidently brings any translation to coincidence with itself.

Any operation of a group must bring the system formed by all the translations of the group to self-coincidence: this is a particular case of the theorem that 'the result of trans-

forming any operation of a group by some other operation of the group itself belongs to the group \*.

§ 3. Two operations are said to be *isomorphic* if one can be derived from the other by multiplying by a translation.

If every operation of a group  $\Gamma$  is isomorphic with some operation of a group  $\Gamma'$  and vice versa, then  $\Gamma$  and  $\Gamma'$  are said to be *isomorphic*.

If  $L, M$  are two isomorphic operations, and  $T$  is a translation such that  $L = M.T$ ; then  $T = M^{-1}.L, M = L.T^{-1}$ . Hence if two of the operations  $L, M, T$  belong to a group, so does the third; therefore:—

*Any series of isomorphic operations of a group is obtained by combining any one of the operations with the translations of the group.*

Suppose then that every operation of *any* group  $\Gamma$  is isomorphic with one of the operations  $A', B', C', D', \dots$  belonging to  $\Gamma$ .

We proved on pp. 151 and 154 that *every* operation is isomorphic with some operation which leaves a point  $P$  (chosen arbitrarily) unmoved, and that the result of combining any number of a series of operations such as  $A', B', C', D', \dots$  is isomorphic with the result of combining the corresponding members of the series  $A, B, C, D, \dots$ ; where  $A, B, C, D, \dots$  are operations leaving the point  $P$  unmoved and isomorphic with  $A', B', C', D', \dots$  respectively. But the result of combining any operations of the series  $A', B', C', D', \dots$  is an operation isomorphic with some member of the series; therefore the result of combining any operations of the series  $A, B, C, D, \dots$  is a member of this series, and hence  $A, B, C, D, \dots$  form a group. This group ( $G$ ) consists solely of operations which leave a point unmoved; it is a *point-group* (see p. 48). We have then:—

*Every group is isomorphic with a point-group.*

It is evident from the above reasoning that those operations of  $\Gamma$  which are isomorphic with the operations of any subgroup of  $G$  form a subgroup of  $\Gamma$ , and vice versa.

§ 4. Let  $L'$  be *any* operation of  $\Gamma$ , and  $L$  the operation of  $G$  isomorphic with  $L'$ ; then  $L' = T.L$  where  $T$  is some translation. Now  $L'$  brings the system of translations ( $\Gamma_+$ ) of  $\Gamma$  into self-coincidence, and so does  $T$ ; therefore  $L$  must do the same.

\* For if  $A$  and  $B$  belong to a group, so does  $A^{-1}.B.A$ .

Hence  $G$  is a group of operations bringing the system  $\Gamma_r$  into self-coincidence.

Now suppose that  $\Gamma$  is a group of the kind considered in § 1 with three finite independent translations and no infinitesimal translations. Such a group will be called a *space-group*. Then the system  $\Gamma_r$  can be represented by a lattice  $\omega$ , one of whose points is the point left unmoved by  $G$ . In this case every operation of  $G$  must bring  $\omega$  to self-coincidence. Hence  $G$  can only contain rotations and rotatory-reflexions through  $2\pi$ ,  $\pi$ ,  $\frac{2\pi}{3}$ ,  $\frac{\pi}{2}$ , or  $\frac{\pi}{3}$  (p. 132); that is,  $G$  is one of the groups discussed in chapters v and vi. Hence:—

*Every space-group is isomorphous with one of the thirty-two point-groups consistent with the law of rational indices, and all the operations of the point-group are symmetry-operations of some lattice representing the system of translations of the space-group.*

It follows that *the angle of any screw or rotatory-reflexion of a space-group is  $2\pi$ ,  $\pi$ ,  $\frac{2\pi}{3}$ ,  $\frac{\pi}{2}$ , or  $\frac{\pi}{3}$ .*

In future we shall mean by 'group' a 'space-group,' unless the contrary is expressly stated.

We see from the above that if  $G$  is a merohedral group contained in any system (triclinic, monoclinic, tetragonal, rhombohedral, hexagonal, or regular), then the symmetry of  $\omega$  must be at least that of the holohedry of the same system.

Thus if  $G$  is  $C_{2v}$ ,  $\Gamma_r$  must be represented by one of the lattices  $\Gamma_o, \Gamma_o', \Gamma_o'', \Gamma_o'''$  (p. 136). It would be consistent with the above reasoning if  $\Gamma_r$  were represented by  $\Gamma_t, \Gamma_t', \Gamma_r, \Gamma_r',$  or  $\Gamma_r''$ ; but in this case the group  $\Gamma$  would only be a specialized form of a group whose translation-group is represented by  $\Gamma_o, \Gamma_o', \Gamma_o'',$  or  $\Gamma_o'''$ , and would not be considered as distinct from it.

We may say in general then that 'the symmetry of  $\omega$  is the holohedry of that system of which  $G$  is a merohedry.' An exception arises in the case of the rhombohedral system; for, though  $\Gamma_h$  has all the symmetry-elements of  $\Gamma_{rh}$ , yet  $\Gamma_h$  is not a specialized form of  $\Gamma_{rh}$ . For the translation-group represented by  $\Gamma_h$  is obtained by combining that represented by  $\Gamma_{rh}$  in the special case in which the translations  $OA_1, OA_2, OA_3$  (Fig. 75) are parallel to a plane with a translation *not belonging to this specialized form of  $\Gamma_{rh}$*  perpendicular to that plane. Thus space-groups isomorphous with a point-group of the rhombohedral system whose translation-groups are

represented by  $\Gamma_h$  are essentially different from those isomorphous with the same point-groups whose translation-groups are represented by  $\Gamma_{rh}$ .

§ 5. Since any symmetry-axis of the lattice  $\omega$  through one of its points passes through a row of its points, and any symmetry-plane (or the plane of any symmetry-operation of the second sort) through one of its points passes through a net of its points, and since the axes or planes of any two isomorphous operations are parallel; therefore any rotation-axis, screw-axis, or axis of rotatory-reflexion of the group  $\Gamma$  is parallel to a translation of  $\Gamma$ , and any symmetry-plane, gliding-plane, or plane of rotatory-reflexion is parallel to two independent translations of  $\Gamma$ .

Let  $C_{\theta,t}$  be a screw of  $\Gamma$ ,  $\theta$  being a positive angle such that no screw of  $\Gamma$  about the same axis has an angle lying between 0 and  $\theta$ . Let  $n\theta = 2\pi$ ; then  $n = 2, 3, 4$ , or  $6$ . Now, since  $C_{\theta,t}$  is an operation of  $\Gamma$ , so is  $(C_{\theta,t})^n = C_{n\theta, nt} = C_{0, nt} =$  a translation  $nt$  parallel to the axis of the screw. Let  $2\tau$  be the primitive translation of  $\Gamma$  parallel to this axis; then  $nt = 2m\tau$  (where  $m$  is an integer), or  $t = \frac{2m\tau}{n}$ .

If  $t$  does not lie between 0 and  $\frac{2(n-1)\tau}{n}$  (inclusive), we can find (by combining  $C_{\theta,t}$  with some multiple of  $2\tau$ ) a screw  $C_{\theta,t'}$  contained in  $\Gamma$  such that  $t'$  lies between these limits, and is therefore one of the quantities  $0, \frac{2\tau}{n}, \frac{4\tau}{n}, \frac{6\tau}{n}, \dots, \frac{2(n-1)\tau}{n}$ .

Such a movement  $C_{\theta,t'}$  is called the *reduced* movement corresponding to the axis of  $C_{\theta,t}$ .

There can only be one reduced movement corresponding to a given axis, for if  $C_{\theta,t'}$ ,  $C_{\theta,t''}$  are two movements belonging to the group, then  $C_{\theta,t'}.(C_{\theta,t''})^{-1}$  belongs to the group and is equivalent to a translation  $t' - t''$  parallel to the axis. But since  $2\tau$  is the *primitive* translation in this direction  $t'$  and  $t''$  must be either equal or must differ by a multiple of  $2\tau$ , and therefore  $t'$  and  $t''$  cannot both be included in the series  $0, \frac{2\tau}{n}, \frac{4\tau}{n}, \frac{6\tau}{n}, \dots, \frac{2(n-1)\tau}{n}$  unless they are equal.

The 'gliding-reflexion'  $S(t)$ , which consists of a reflexion in a plane  $\sigma$  combined with a translation  $t$  parallel to that plane, has similar properties.

Since the reflexion and translation of  $S(t)$  are permutable

(p. 150, note), and the square of a simple reflexion = 1, therefore  $\{S(t)\}^2$  is equivalent to a translation  $2t$  parallel to the plane  $\sigma$ . If then  $S(t)$  belongs to a group of movements  $\Gamma$  so does the translation  $2t$ .

Let  $2\tau_1, 2\tau_2$  be a primitive pair of translations of  $\Gamma$  parallel to  $\sigma$ . Then  $2t = 2p_1\tau_1 + 2p_2\tau_2$  where  $p_1$  and  $p_2$  are integers, or  $t = p_1\tau_1 + p_2\tau_2$ .

Combining  $S(t)$  with multiples of  $2\tau_1$  and  $2\tau_2$  we see that there is always some operation  $S(t')$  belonging to  $\Gamma$  such that  $t'$  is one of the quantities  $0, \tau_1, \tau_2, \tau_1 + \tau_2$ . Such an operation is called the *reduced operation* corresponding to the pair  $2\tau_1, 2\tau_2$ . As in the case of a screw, it may be readily shown that there is only *one* reduced operation corresponding to each primitive pair in the plane  $\sigma$ .

§ 6. Let  $A_{\theta, t_a}$  be a screw of a group  $\Gamma$  about an axis  $a$ , and let  $T$  be any translation of  $\Gamma$  whose components parallel and perpendicular to  $a$  are  $\tau_a, \tau_n$ . Then  $A_{\theta, t_a} \cdot T =$  a screw  $B_{\theta, t_a + \tau_a}$  about some axis  $b$  parallel to  $a$ . Thus the 'reduced' movements corresponding to the axes  $a$  and  $b$  have the same angle, and have or have not the same translation according as  $\tau_a$  is or is not a multiple of the primitive translation of  $\Gamma$  parallel to  $a$ . Hence:—

*All reduced screws of a space-group which are isomorphous with a single rotation of the corresponding point-group have or have not the same translation according as the space-group does not or does contain a translation whose component parallel to the axes of the screws is less than the primitive translation of the space-group in this direction.*

Consider again a set of screws contained in  $\Gamma$  and isomorphous with a single operation of the corresponding point-group. We may take as a primitive triplet of the translations  $2\tau, 2\tau_1, 2\tau_2$  where  $2\tau$  is the primitive translation parallel to the axes of the screws. Let  $e$  be the axis of any one of the screws, and let  $e_1, e_2, e_3$  be the positions into which  $e$  is brought by the translations  $2\tau_1, 2\tau_2, 2\tau_1 + 2\tau_2$ . Let  $e, e_1, e_2, e_3$  be cut in the points  $E, E_1, E_2, E_3$  by a plane perpendicular to them all. Let the axis of any one of the isomorphous set of screws be  $f$ ; then it is evident that there must be a similar screw round some axis which meets the surface or circumference of the parallelogram  $E E_1 E_2 E_3$  and which is obtained from  $f$  by transforming it by some translation of the form  $2p_1\tau_1 + 2p_2\tau_2$  (where  $p_1$  and  $p_2$  are integers); hence:—

*To find all the screws isomorphous with a single rotation*

of the corresponding point-group it is only necessary to find all whose axes meet a certain parallelogram, and to transform them by the translations of the group.

§ 7. We shall now prove certain theorems by means of which one group can be derived from another. Suppose the operations of a point-group  $G_1$  are

$$1, L_1, L_2, L_3, \dots$$

Let  $H$  be an operation bringing the system of symmetry-elements of  $G_1$  to self-coincidence; then  $H^2, H^3, \dots$  also bring this system to self-coincidence.

Since the system can be thus brought to self-coincidence in only a finite number of ways some power of  $H$  is unity\*, that is, some power of  $H$  is an operation of  $G$ . Let  $H^m (= L_h)$  be the lowest power of  $H$  which is an operation of  $G$ .

$$\text{Then } \left. \begin{array}{ccccccc} 1, & L_1, & L_2, & L_3, & \dots \\ H, & H.L_1, & H.L_2, & H.L_3, & \dots \\ H^2, & H^2.L_1, & H^2.L_2, & H^2.L_3, & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ H^{m-1}, & H^{m-1}.L_1, & H^{m-1}.L_2, & H^{m-1}.L_3, & \dots \end{array} \right\} \quad (i)$$

form a group  $G$ .

For the product of any two of these operations  $H^a.L_p$  and  $H^b.L_q$  (say)  $= H^a.L_p.H^b.L_q = H^{a+\beta}.H^{-\beta}.L_p.H^b.L_q$ .

Now, since  $H^b$  brings the system of symmetry-elements of  $G_1$  into self-coincidence, the result of transforming any operation of  $G_1$  by  $H^b$  is another operation of  $G_1$  (p. 156).

Therefore the product of  $H^a.L_p$  and  $H^b.L_q = H^{a+\beta}.L_q = H^{a+\beta}.L_u = H^{a+\beta-m}.L_h.L_u = H^{a+\beta-m}.L_k$ . Hence the product is a member of the series (i); for either  $a+\beta$  or  $a+\beta-m$  is an integer lying between 0 and  $(m-1)$  inclusive.

It is readily seen that no two members of the series (i) are identical. For if  $H^a.L_p = H^b.L_q$ ,  $H^{a-\beta} = L_q.L_p^{-1} = L_w$ , which is only possible if  $a = \beta$  and  $L_w = 1$ .

We may show that the operations obtained by multiplying  $1, L_1, L_2, L_3, \dots$  by  $1, H, H^2, \dots, H^{m-1}$  on the right-hand side form the same group. For  $L_v.H^\gamma = H^\gamma.H^{-\gamma}.L_v.H^\gamma = H^\gamma.L_s$ .

The case of greatest importance is that for which  $m = 2$ ; the theorem then becomes:—

'If  $1, L_1, L_2, L_3, \dots$  are the operations of a group  $G_1$ , and  $H$  is an operation which brings the system of symmetry-elements

\* The angle of the operation  $H$  is assumed finite.

of  $G_1$  to self-coincidence and is such that  $H^2$  is an operation of  $G_1$ , then

$$\left. \begin{matrix} 1, & L_1, & L_2, & L_3, & \dots \\ H, & H.L_1, & H.L_2, & H.L_3, & \dots \end{matrix} \right\} \text{ form a group.}$$

In chapter vi we used the equally sound but longer process of verifying this fact in each particular case.

§ 8. If  $V$  is any space- or point-group and  $K$  any operation, we shall in future express by the symbols  $[K, V]$ ,  $[V, K]$  the series of operations formed by multiplying all the operations of  $V$  by  $K$  on the left- or right-hand side respectively.

Now let  $\Gamma_1$  be a space-group isomorphous with  $G_1$  (of § 7), and let  $L'_e$  be any operation of  $\Gamma_1$  isomorphous with  $L_e$ ; then the series of operations of  $\Gamma_1$  which are isomorphous with  $L_e$  is  $[L'_e, \Gamma_\tau]$ , where  $\Gamma_\tau$  is the group of translations of  $\Gamma_1$  (p. 158).

Let now  $H'$  be an operation isomorphous with  $H$  which brings the system of symmetry-elements of  $\Gamma_1$  to self-coincidence.

Then  $H'^m$  is isomorphous with some operation of  $G_1$ ; take  $H'$  such that  $H'^m$  is an operation of  $\Gamma_1$  ( $= L'_h$  say).

We can prove now that the series of operations

$$\left. \begin{matrix} \Gamma_\tau, & [L'_1, \Gamma_\tau], & [L'_2, \Gamma_\tau], & [L'_3, \Gamma_\tau], & \dots \\ [H', \Gamma_\tau], & [H'.L'_1, \Gamma_\tau], & [H'.L'_2, \Gamma_\tau], & [H'.L'_3, \Gamma_\tau], & \dots \\ [H'^{m-1}, \Gamma_\tau], & [H'^{m-1}.L'_1, \Gamma_\tau], & [H'^{m-1}.L'_2, \Gamma_\tau], & [H'^{m-1}.L'_3, \Gamma_\tau], & \dots \end{matrix} \right\} \text{(ii)}$$

each of which is isomorphous with some operation of (i), forms a group  $\Gamma$  isomorphous with  $G$ .

For the product of any two of the operations (ii), e. g.  $H'^a.L'_{p_1}.T_1$  and  $H'^\beta.L'_q.T_2 = H'^{a+\beta}.L'_{p_1}.H'^\beta.L'_q.T_2$  (where  $L'_{p_1} = L'_{p_1}.T_1 = H'^{a+\beta}.L'_n.T_3 = H'^{a+\beta-m}.L'_h.L'_n.T_3 = H'^{a+\beta-m}.L'_k.T_4$  (where  $L'_n, L'_k$ , &c., are members of the first row of (ii) and  $T_1, T_2, T_3, T_4$  some translations contained in  $\Gamma_\tau$ ) = some member of the series (ii), since either  $a+\beta$  or  $a+\beta-m$  is an integer lying between 0 and  $(m-1)$  inclusive. It is readily seen that no two members of the series (ii) are identical.

Conversely every group  $\Gamma$  isomorphous with  $G$  can be obtained by multiplying the operations of the subgroup  $\Gamma_1$  consisting of those operations of  $\Gamma$  which are isomorphous with the operations of  $G_1$  by some operation  $H'$  which is isomorphous with  $H$ , brings the system of symmetry-elements of  $\Gamma_1$  into self-coincidence, and is such that the lowest power of  $H'$

which is isomorphous with an operation of  $\Gamma_1$  is an operation of  $\Gamma_1$ .

For let  $H'$  be any operation of  $\Gamma$  isomorphous with  $H$ . Then, firstly,  $H'$  is a symmetry-operation of the axes and planes of  $\Gamma_1$ . For  $H'^{-1} \cdot L'_j \cdot H'$  (where  $L'_j$  is any operation of  $\Gamma_1$ ) is an operation of  $\Gamma$  and is isomorphous with  $H^{-1} \cdot L_j \cdot H$ , that is, with some operation of  $G_1$ ; therefore  $H'^{-1} \cdot L'_j \cdot H'$  is an operation of  $\Gamma_1$ . Hence  $H'$  is a symmetry-operation of the system of elements of  $\Gamma_1$ .

Secondly, the lowest power of  $H'$  which is isomorphous with an operation of  $\Gamma_1$  is an operation of  $\Gamma$  and is therefore an operation of  $\Gamma_1$ .

Thirdly, the group formed by combining the operations of  $\Gamma_1$  with  $H'$  is identical with  $\Gamma$ . For since by combining the operations of  $G_1$  with  $H$  we get every operation of  $G$ , therefore by combining some operation of  $\Gamma_1$  with  $H'$  we get an operation of  $\Gamma$  isomorphous with any given operation of  $\Gamma$ . But all isomorphous operations of  $\Gamma$  are obtained by multiplying any one of them by the operations of the translation-group ( $\Gamma_r$ ) of  $\Gamma$ . Now evidently any space-group which is a subgroup of  $\Gamma$  must have the same translation-group as  $\Gamma$ ; so that the operations of  $\Gamma_r$  are included in those of  $\Gamma_1$ . We can therefore get every operation of  $\Gamma$  by combining  $H'$  with operations of  $\Gamma_1$ . Hence we have the following theorem:—

*A point-group  $G$  is formed by combining the operations of a subgroup  $G_1$  with an operation  $H$  which brings the system of symmetry-elements of  $G_1$  to self-coincidence.  $\Gamma_1$  is any space-group isomorphous with  $G_1$ ; and  $H'$  is any operation isomorphous with  $H$  which brings the system of symmetry-elements of  $\Gamma_1$  to self-coincidence, and is such that the lowest power of  $H'$  which is isomorphous with an operation of  $\Gamma_1$  is itself an operation of  $\Gamma_1$ \*. Then a group isomorphous with  $G$  is obtained by combining the operations of  $\Gamma_1$  with  $H'$ †; and conversely all space-groups isomorphous with  $G$  can be obtained in a similar way.*

This theorem is of fundamental importance.

\* This last condition is always fulfilled if  $H'$  is an inversion or a simple reflexion. When  $H'$  is a gliding-reflexion  $S(t)$  in a plane which is not parallel to any symmetry plane of  $G_1$ , the condition is fulfilled if and only if  $2t$  is a translation of  $\Gamma_1$ . When  $H'$  is a screw  $A_{2\pi/n}$ , about an axis which is not parallel to any axis of  $G_1$ , the condition is fulfilled if and only if  $n\pi$  is a translation of  $\Gamma_1$ .

† The groups obtained by taking various operations such as  $H'$  are not necessarily all distinct. Groups whose subgroups  $\Gamma_1$  are distinct are evidently themselves distinct.

The operations of the series (ii) are all different; the group formed by them will be denoted by the symbol  $\{H', \Gamma_1\}$ .

§ 9. Let  $H'$  be an operation of the second sort which brings the system of symmetry-elements of a group  $\Gamma_1$  into self-coincidence. Suppose an axis  $a$  to be brought into coincidence with an axis  $b$  by  $H'$ .

Now the result of transforming a right-handed screw by an operation of the second sort is a similar left-handed screw\* (p. 156). Hence if  $b$  is identical with  $a$ , the group must contain similar right-handed and left-handed screws about  $a$ . This is only possible if the translation of these screws is either zero or half a translation of the group parallel to  $a$ . If however  $b$  is not identical with  $a$ ,  $\Gamma_1$  is such that to every right-handed screw about any axis there corresponds a similar left-handed screw about some other axis. Hence:—

*If the system of elements of a group  $\Gamma_1$  is brought to self-coincidence by an operation of the second sort, either the axes of  $\Gamma_1$  are rotation-axes and screw-axes whose translation is half a translation of  $\Gamma_1$  in the direction of the axes, or else similar right-handed and left-handed screws of  $\Gamma_1$  occur in pairs.*

This theorem is also of great importance: it shows that no space-group  $\Gamma$  can be obtained by combining an operation of the second sort with the operations of any group  $\Gamma_1$  whose symmetry-elements do not satisfy the above condition.

§ 10. Let  $H_1', H_2'$  be two symmetry-operations of the system of elements of any group  $\Gamma_1$ ; let  $H_1'^m, H_2'^n$  be the lowest powers of  $H_1'$  and  $H_2'$  respectively, which are isomorphous with an operation of  $\Gamma_1$ , and let  $H_1'^m, H_2'^n$  be operations of  $\Gamma_1$ .

Then if  $\{H_1', \Gamma_1\}$  and  $\{H_2', \Gamma_1\}$  are identical † ( $= \Gamma$ ),  $\Gamma$  contains both  $H_1'$  and  $H_2'$ ; moreover, since in this case  $\{H_1', \Gamma_1\}$  and  $\{H_2', \Gamma_1\}$  must contain the same number of distinct sets of isomorphous operations, we must have  $m = n$  (cf. p. 163).

Again if  $\{H_1', \Gamma_1\}$  contains  $H_2'$  and if  $m$  is prime,  $\{H_1', \Gamma_1\}$  and  $\{H_2', \Gamma_1\}$  are identical and  $m = n$ .

For if  $\{H_1', \Gamma_1\}$  contains  $H_2', H_1'^r \cdot L' = H_2'$  where  $L'$  is some operation of  $\Gamma_1$  and  $r$  some positive integer  $< m$  (p. 163).

\* That is, a screw of the same translation and of equal and opposite angle of rotation.

† This identity of the two groups is expressed by the symbol  $=$ . Thus  $\{H_1', \Gamma_1\} = \{H_2', \Gamma_1\}$ .

Therefore  $\{H_2', \Gamma_1\} = \{H_1'^r \cdot L', \Gamma_1\} = \{H_1'^r, \Gamma_1\}$ .

Now since  $r$  is prime to  $m$  the series

$$\Gamma_1, [H_1'^r, \Gamma_1], [H_1'^{2r}, \Gamma_1], \dots, [H_1'^{(m-1)r}, \Gamma_1]$$

is evidently identical with

$$\Gamma_1, [H_1', \Gamma_1], [H_1'^2, \Gamma_1], \dots, [H_1'^{(m-1)}, \Gamma_1]^*;$$

for  $0, r, 2r, \dots, (m-1)r$  leave different remainders when divided by  $m$ .

Hence  $\{H_1', \Gamma_1\} = \{H_1'^r, \Gamma_1\} = \{H_2', \Gamma_1\}$ , and therefore  $m = n$ .

Similarly if  $\{H_2', \Gamma_1\}$  contains  $H_1'$  and  $n$  is prime,  $\{H_2', \Gamma_1\}$  and  $\{H_1', \Gamma_1\}$  are identical and  $n = m$ .

The case of  $m = n = 2$  is of special importance.

§ 11. Let a space-group isomorphous with a point-group  $G$  have two subgroups  $\Gamma_1, \Gamma_2$  isomorphous with the point-groups  $G_1, G_2$  respectively. Suppose that *every* operation of  $G$  is of the form  $M \cdot N$ , where  $M$  is an operation of  $G_1$  and  $N$  an operation of  $G_2$ . Then evidently *every* operation of  $\Gamma$  is of the form  $M' \cdot N'$ , where  $M'$  is an operation of  $\Gamma_1$ , and  $N'$  an operation of  $\Gamma_2$ .

Let  $H'$  be a symmetry-operation of the elements of  $\Gamma_1$  and the elements of  $\Gamma_2$ . Then

$$H'^{-1} \cdot M' \cdot H' = M_1', \quad H'^{-1} \cdot N' \cdot H' = N_1',$$

where  $M_1'$  is an operation of  $\Gamma_1$  and  $N_1'$  is an operation of  $\Gamma_2$ .

Therefore the result of transforming any operation  $M' \cdot N'$  of  $\Gamma$  by  $H'$  is  $H'^{-1} \cdot M' \cdot N' \cdot H' = H'^{-1} \cdot M' \cdot H' \cdot H'^{-1} \cdot N' \cdot H' = M_1' \cdot N_1'$  which is another operation of  $\Gamma$ .

Therefore  $H'$  is a symmetry-operation of the elements of  $\Gamma$ .

§ 12. If we subject *any* point  $P$  to all the movements of a space-group  $\Gamma$  isomorphous with a point-group  $G$ , we obtain an infinite series of 'equivalent' points. There is no difficulty in obtaining the coordinates of all the points of such a series referred to any axes whose origin is the centre of  $G$ . Take one operation of  $\Gamma$  isomorphous with each operation  $L_1, L_2, L_3, \dots$  of  $G$ ; suppose these to be  $L_1', L_2', L_3', \dots$ . Let  $L_{i_n}'$  be *any* operation of  $\Gamma$  isomorphous with  $L_i'$ ; then  $L_{i_n}' = L_i' \cdot T$  where  $T$  is some translation of  $\Gamma$ , and  $L_i' = L_i \cdot T_1$  where  $T_1$  is some translation which may readily be obtained in each particular case. Then any point  $P$  is brought by  $L_i$  to a point whose coordinates are known from chapter vii; adding on to these

\* The operations are not in general arranged in the same order in the two series.

coordinates the components of  $T_1$  along the axes we obtain the coordinates of the point to which  $P$  is brought by  $L'_i$ . Similarly we obtain the coordinates of the points to which  $P$  is brought by each of the operations  $L'_1, L'_2, L'_3, \dots$ . To obtain the coordinates of all the points to which  $P$  is brought by the operations of  $\Gamma$  we have only to add to these the components of all the translations of  $\Gamma$ .

For the application of this method to each one of the 230 possible space-groups we must refer the reader to Schoenflies' "Krystallsysteme und Krystallstructur."

§ 13. We are at last in a position to find all possible space-groups which have a crystallographic application. All those which are isomorphous with a given point-group form a *class* which is named after the point-group. A space-group is said to belong to the same *system* as the isomorphous point-group.

We shall denote any space-group by the same symbol as the isomorphous point-group with the addition of a positive integral index.

Thus, for example,  $C_{4v}^m$  denotes 'any\* space-group isomorphous with the point-group  $C_{4v}$ ' or 'any\* space-group of the hemimorphic hemihedry of the tetragonal system'; while  $C_{4v}^1, C_{4v}^2, C_{4v}^3, \dots$  denote *definite* space-groups isomorphous with  $C_{4v}$ .

A group is completely given when the arrangement of all its symmetry-elements (i. e. its screw-axes, gliding-planes, &c.) relatively to its translations is given.

\* Or 'some space-group,' 'a space-group,' &c.

## CHAPTER XVIII

## TRICLINIC AND MONOCLINIC GROUPS.

## § 1. TRICLINIC HEMIHEDRY.

A group of this class can only have operations isomorphous with unity, since this is the only operation of the isomorphous point-group  $C_1$ ; that is, it can only have translations. Considering for this purpose the fourteen translation-groups of chapter xiv as merely specialized forms of the most general of them  $(\Gamma_{tr})^*$ , we may say that *there is only one group of the triclinic hemihedral class*. We denote it by  $C_1^1$  or  $\Gamma_{tr}$ .

## § 2. TRICLINIC HOLOHEDRY.

Since the point-group  $C_i$  is obtained by multiplying the operation of  $C_1$  by an inversion, therefore all groups  $C_i^m$  isomorphous with  $C_i$  are obtained by multiplying the operations of  $\Gamma_{tr}$  by an operation isomorphous with an inversion, that is, by an inversion †. Now *every* group of translations is brought to self-coincidence by an inversion about *any* point  $P$ ; there is therefore one and only one distinct group of this class, namely

$$C_i^1 = \{\Gamma_{tr}, I\}.$$

We may represent the translations of  $C_i^1$  by a lattice, one of whose points is at  $P$ ; let  $Q$  be any other point of the lattice; then an inversion about  $P$  followed by a translation represented by  $PQ$  is equivalent to an inversion about a point bisecting the line  $PQ$  (p. 155). Hence the symmetry-elements of  $C_i^1$  are a series of centres of symmetry at the points of a lattice representing  $\Gamma_{tr}$  and at the centres of all straight lines joining two such points ‡.

\* We shall in future denote a translation-group by the same symbol as the lattice representing it.

† We pointed out on p. 155 that every operation isomorphous with an inversion is an inversion.

‡ That is, at the vertices, centres, centres of the faces, and centres of the edges of the parallelepipeda of Fig. 66.

## § 3. MONOCLINIC HEMIHEDRY.

A group of the monoclinic system has either  $\Gamma_m$  or  $\Gamma'_m$  as its translation-group (p. 133).

Since the point-group  $C_2$  is obtained by multiplying the operations of  $C_1$  by a reflexion, therefore groups  $C_s^m$  isomorphous with  $C_2$  are obtained by combining the operations of  $C_1^1$  with a gliding-reflexion  $S(t)$  in some plane  $\sigma$  (where  $t$  is zero or half a translation of  $C_1^1$  in the plane  $\sigma$ ), which brings the system of translations  $C_1^1$  to self-coincidence. This last condition is only satisfied if  $C_1^1$  has the specialized form  $\Gamma_m$  or  $\Gamma'_m$ , and if  $\sigma$  is perpendicular to the translation  $2\tau_z$  of  $\Gamma_m$  or  $\Gamma'_m$  (using the notation of p. 133).

Suppose now  $C_1^1$  identical with  $\Gamma_m$ .

First, take  $t = 0$ . We have then a group,

$$C_s^1 = \{\Gamma_m, S\}.$$

Since  $S$  followed by the translation  $2p\tau_z$  is equivalent to a reflexion in a plane parallel to  $\sigma$  and at a distance  $p\tau_z$  from it,  $C_s^1$  has a 'set' of symmetry-planes of interval  $\tau_z$  (p. 115).

Secondly, take  $t \neq 0$ . We have then a group

$$C_s^2 = \{\Gamma_m, S(t)\},$$

$2t$  is a translation parallel to  $\sigma$ ; two groups  $\{\Gamma_m, S(t_1)\}$ ,  $\{\Gamma_m, S(t_2)\}$  are not geometrically distinguishable unless  $t_1$  or  $t_2$  (not both)  $= 0$ .

$S(t)$  followed by the translation  $2p\tau_z$  is evidently equivalent to an operation exactly similar to  $S(t)$  in a plane whose distance from  $\sigma$  is  $p\tau_z$ ; hence  $C_s^2$  has a set of similar gliding-planes of interval  $\tau_z$ .

Since  $\Gamma_m$  has no translation whose component perpendicular to  $\sigma$  is not a multiple of  $2\tau_z$ , therefore  $C_s^1$  and  $C_s^2$  have no symmetry-elements other than those mentioned.

§ 4. Next consider the result of combining the translation-group  $\Gamma'_m$  with  $S(t)$ .

A primitive triplet of  $\Gamma'_m$  may be taken as  $2\tau_e$ ,  $2\tau_f$ ,  $\tau_e + \tau_f + \tau_z$ , where  $2\tau_e$ ,  $2\tau_f$  are a primitive pair in the plane perpendicular to the symmetry-axis of  $\Gamma'_m$  (and therefore parallel to  $\sigma$ ) and  $2\tau_z$  is the primitive translation perpendicular to the plane\*.

We need only consider the *reduced* operation  $S(t)$ ; it is of the form  $S(a_1\tau_e + a_2\tau_f)$ , where  $a_1, a_2$  have any one of the sets of values 0, 0; 0, 1; 1, 0; 1, 1.

Now any operation of  $\Gamma'_m$  is of the form  $2p_1\tau_e + 2p_2\tau_f + p_3(\tau_e + \tau_f + \tau_z)$ . If  $p_3$  is even the result of combining this with

\* We still use the notation of p. 133.

$S(a_1\tau_e + a_2\tau_f)$  and the translation  $-(2p_1 + p_3)\tau_e - (2p_2 + p_3)\tau_f$  is an operation  $S(a_1\tau_e + a_2\tau_f)$  in a plane whose distance from  $\sigma$  is  $\frac{p_3}{2}\tau_z$ .

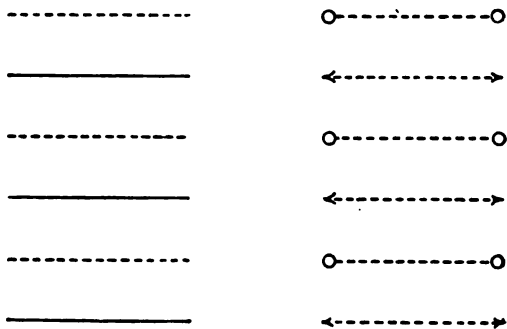
The group  $\{\Gamma'_m, S(a_1\tau_e + a_2\tau_f)\}$  therefore contains a series of operations similar to  $S(a_1\tau_e + a_2\tau_f)$  in a 'set' of planes of interval  $\tau_z$ .

If  $p_3$  is odd the result of combining  $2p_1\tau_e + 2p_2\tau_f + p_3(\tau_e + \tau_f + \tau_z)$  with  $S(a_1\tau_e + a_2\tau_f)$  and the translation  $-(2p_1 + p_3 - 1)\tau_e - (2p_2 + p_3 - 1)\tau_f$  is an operation  $S(a_1 + 1\tau_e + a_2 + 1\tau_f)$  in a plane whose distance from  $\sigma$  is  $\frac{p_3}{2}\tau_z$ .

The group  $\{\Gamma'_m, S(a_1\tau_e + a_2\tau_f)\}$  therefore contains a series of operations of the type  $S(a_1 + 1\tau_e + a_2 + 1\tau_f)$  in a set of planes of interval  $\tau_z$ , which lie halfway between the gliding-planes whose operation is  $S(a_1\tau_e + a_2\tau_f)$ .

Hence, if  $a_1 = a_2 = 0$ , the group  $\{\Gamma'_m, S(a_1\tau_e + a_2\tau_f)\}$  has a series of operations  $S$  in a set of planes of interval  $\tau_z$ , and a series of operations  $S(\tau_e + \tau_f)$  in the planes halfway between them.

If  $a_1 = a_2 = 1$ , the group has a series of operations  $S(\tau_e + \tau_f)$  in a set of planes of interval  $\tau_z$ , and a series of operations  $S(2\tau_e + 2\tau_f)$ , and therefore a series of operations  $S^*$  in the planes halfway between them.

Fig. 84.  $C_s^3$ .Fig. 85.  $C_s^4$ .

These two cases are not geometrically distinguishable, we denote them both by

$$C_s^3 = \{\Gamma'_m, S\}.$$

\* For  $2\tau_e + 2\tau_f$  is a translation of the group.

If  $a_1 = 0$ ,  $a_2 = 1$ , the group has a series of operations  $S(\tau_f)$  in a set of planes of interval  $\tau_z$ , and a series of operations  $S(\tau_e + 2\tau_f)$ , and therefore a series of operations  $S(\tau_e)^*$  in the planes halfway between them.

The case of  $a_1 = 1$ ,  $a_2 = 0$  is of course not geometrically distinguishable from this; we denote the group in this case by

$$C_s^4 = \{\Gamma'_m, S(\tau)\}.$$

We may represent the groups  $C_s^3$ ,  $C_s^4$  diagrammatically in Figs. 84 and 85, if we take the symmetry-axis of  $\Gamma'_m$  vertical. If the plane of an operation  $S(t)$  is perpendicular to the plane of the paper, its intersection with the paper will be denoted † by a simple line if  $t = 0$ ; by a broken line if the direction of  $t$  ( $\neq 0$ ) is perpendicular to the plane of the paper; by a broken line with arrows if the direction of  $t$  is in the plane of the paper, and by a broken line with small circles if  $t$  has any other direction.

In Fig. 84 we have taken the direction of  $\tau_e + \tau_f$  perpendicular to the plane of the paper.  $C_s^1$  would be represented by the simple lines of Fig. 84 taken alone,  $C_s^2$  by the broken lines. In Fig. 85 we have taken the direction of  $\tau_e$  parallel to the plane of the paper.  $C_s^2$  would be represented either by the broken lines alone, or by the broken lines with circles alone.

The distance between the extreme lines of either figure is  $\frac{5\tau_z}{2}$ .

### § 5. MONOCLINIC HEMIMORPHY.

Since the point-group  $C_2$  is obtained by combining the operation of  $C_1$  with a rotation through  $\pi$  about some axis, therefore the groups  $C_2^m$  are obtained by combining a translation-group with some screw  $A_{\pi, t}$  which brings the system of translations of self-coincidence. To fulfil this last condition the translation-group must be either  $\Gamma_m$  or  $\Gamma'_m$  (cf. p. 159), and the axis of the screw ( $e$ ) must be parallel to  $\tau_z$ .

We may take the screw as a reduced operation; in this case  $t = 0$  or  $\tau_z$  (p. 160).

First consider the result of combining  $A_{\pi, t}$  with  $\Gamma_m$  of which  $2\tau_e$ ,  $2\tau_f$ ,  $2\tau_z$  are a primitive triplet.

Since there is no translation whose component in the direction of  $e$  is not a multiple of  $2\tau_z$ , the reduced screws about all axes parallel to  $e$  have the same translation (p. 161); that is, the group  $\{\Gamma_m, A_{\pi, t}\}$  contains only rotation-

\* For  $2\tau_f$  is a translation of the group.

† This is Fedorow's method.

axes or only screw-axes, according as  $t=0$  or  $\tau_z$ . To find the distribution of these axes, suppose that  $e$  is perpendicular to the plane of the paper (Fig. 86) and meets it in  $E$ , and let  $EE_1$ ,  $EE_2$ ,  $EE_3$  represent  $2\tau_e$ ,  $2\tau_f$ ,  $2\tau_e + 2\tau_f$  respectively.

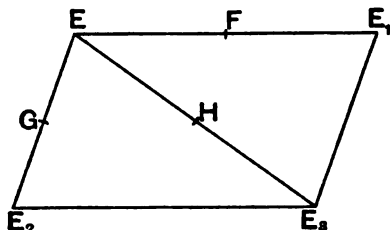
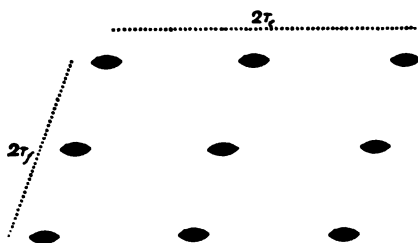
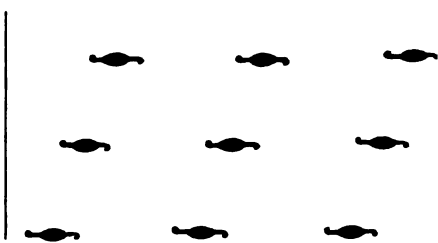


Fig. 86.

Then, since the translations bring the system of symmetry-axes to self-coincidence, axes similar to  $e$  pass through  $E_1, E_2, E_3$ . To find the whole system of axes it is only necessary to find those meeting the surface or circumference of the parallelogram  $EE_1E_2E_3$ , and to transform them by

the translations of the group (p. 161). Suppose such an axis meets the parallelogram in  $P$ , then the screw about  $e$ , followed by the screw about a parallel axis through  $P$ , is equivalent to a translation whose component parallel to the plane of the

Fig. 87.  $C_2^1$ .Fig. 88.  $C_2^2$ .

parallelogram is  $2EP$ . Therefore  $P$  must bisect a side or diagonal of the parallelogram.

We have then two groups,

$$C_2^1 = \{\Gamma_m, A_{\pi, 0}\}, C_2^2 = \{\Gamma_m, A_{\pi, \tau_z}\},$$

the arrangement of whose axes is shown in Figs. 87 and 88\*, if we represent a 2-al rotation-axis perpendicular to the plane of the paper by  $\bullet$  and a similar screw-axis by  $\curvearrowright$ .

§ 6. Next consider the result of combining  $A_{\pi, t}$  with  $\Gamma'_m$ , of which  $2\tau_z$ ,  $\tau_e + \tau_z$ ,  $\tau_f + \tau_z$  are a primitive triplet.

\* In these and all other similar figures we shall suppose ourselves looking from the positive towards the negative end of the axes; that is, the translation  $\tau_z$  brings an object closer to us.

As before, let  $e$  be perpendicular to the plane of the paper and meet it in  $E$ ; let  $EE_1$ ,  $EE_2$ ,  $EE_3$  represent  $\tau_e$ ,  $\tau_f$ ,  $\tau_e + \tau_f$  respectively.

Then, since the result of transforming  $A_{\pi, t}$  by  $\tau_e + \tau_z$ ,  $\tau_f + \tau_z$ , or  $(\tau_e + \tau_z) + (\tau_f + \tau_z)$  is a similar operation of the group, axes exactly similar to  $e$  must pass through  $E_1$ ,  $E_2$ ,  $E_3$ . As before, we find all the axes by transforming those which meet the surface and circumference of  $EE_1E_2E_3$  by the operations of  $\Gamma'_m$ ; and, as before, we see that all axes meeting this parallelogram must bisect a side or diagonal. Now  $A_{\pi, t}$  followed by the translation  $\tau_e + \tau_z$  is equivalent to a screw of angle  $\pi$  and translation  $t + \tau_z$  about the axis  $f$  parallel to  $e$  which bisects  $EE_1$ ; and  $A_{\pi, t}$  followed by the translation  $(\tau_e + \tau_z) + (\tau_f + \tau_z)$  and the translation  $-\tau_z$  is equivalent to a screw of angle  $\pi$  and translation  $t$  about the axis  $h$  parallel to  $e$  which bisects  $EE_3$ . Hence, if  $t = 0$ , we have rotation-axes through the vertices and centre of  $EE_1E_2E_3$ , and screw-axes of translation  $\tau_z$  through the middle points of the sides.

The general arrangements of the axes of the group

$$C_2^3 = \{\Gamma'_m, A_{\pi, 0}\}$$

is shown in Fig. 89, from which also it is evident that the group  $\{\Gamma'_m, A_{\pi, \tau_z}\}$ , which has rotation-axes through  $F$  and  $G$ , and screw-axes through  $E$  and  $H$ , is geometrically identical with  $C_2^3$ .

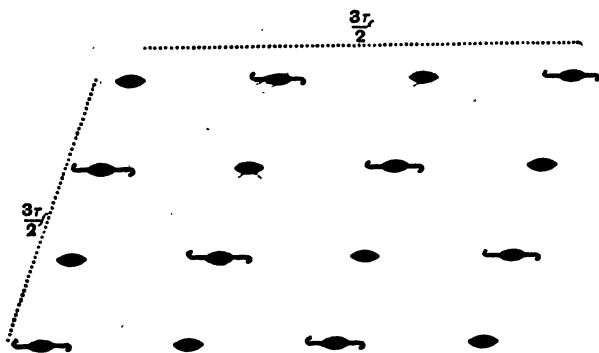


Fig. 89.  $C_2^3$ .

If we had taken  $E_1E = \tau_f$ ,  $EE_3 = \tau_e$ ,  $EE_2 = \tau_e + \tau_f$  (instead of  $EE_1 = \tau_e$ ,  $EE_2 = \tau_f$ ,  $EE_3 = \tau_e + \tau_f$ ), we should again have an arrangement of axes similar to that of Fig. 86; but

now the axes of  $\{\Gamma'_m, A_{\pi,0}\}$  through  $E$  and  $G$  would be rotation-axes, and those through  $F$  and  $H$  screw-axes, that is, the rotation-axes would have passed through *adjacent* (instead of *opposite*) vertices of the parallelogram  $EFGH$ .

### § 7. MONOCLINIC HOLOHEDRY.

Since the point-group  $C_{2h}$  is obtained by multiplying the operations of  $C_2$  by an inversion, therefore a group  $C_{2h}^m$  is obtained by multiplying any group  $C_2^m$ , which fulfils the conditions of p. 165, by an operation isomorphous with an inversion (i.e. by an inversion) which brings the system of symmetry-elements of  $C_2^m$  into self-coincidence; and all groups  $C_{2h}^m$  can be so obtained (p. 164).

All three groups,  $C_2^1, C_2^2, C_2^3$ , satisfy the condition of p. 165; their systems of symmetry-elements are evidently brought to self-coincidence by inversion about a point  $P$  if, and only if,  $P$  lies either in an axis, or halfway between two axes of the same sort (i.e. both rotation- or both screw-axes).

First, suppose  $P$  lies in an axis; we have then three groups

$$C_{2h}^1 = \{C_2^1, I\} = \{C_2^1, S\}. \quad \text{Subgroup } C_s^1.$$

$$C_{2h}^2 = \{C_2^2, I\} = \{C_2^2, S\}. \quad \text{Subgroup } C_s^1.$$

$$C_{2h}^3 = \{C_2^3, I\} = \{C_2^3, S\}. \quad \text{Subgroup } C_s^3.$$

Since  $C_{2h}$  has a subgroup  $C_s$ , every group  $C_{2h}^m$  has a subgroup  $C_s^m$ .

Now the operation  $A_{\pi,e}$  about an axis  $e$ , followed by an inversion about a point in  $e$  is equivalent to a reflexion in a plane perpendicular to  $e$ . Therefore  $C_s^1$  is the subgroup of  $C_{2h}^1, C_{2h}^2$ , and  $C_s^3$  that of  $C_{2h}^3$ . We could have derived  $C_{2h}^1, C_{2h}^2, C_{2h}^3$  from  $C_2^1, C_2^2, C_2^3$  respectively, by combining their operations with a reflexion instead of with an inversion, if we had so chosen\*.

Next, suppose  $P$  lies halfway between two axes of the same sort; we have then these three groups

$$C_{2h}^1 = \{C_2^1, I_1\} = \{C_2^1, S(t)\}. \quad \text{Subgroup } C_s^2.$$

$$C_{2h}^2 = \{C_2^2, I_1\} = \{C_2^2, S(t)\}. \quad \text{Subgroup } C_s^2.$$

$$C_{2h}^3 = \{C_2^3, I_1\} = \{C_2^3, S(t)\}. \quad \text{Subgroup } C_s^4.$$

These three groups contain no simple reflexions (for they

\* Since  $\{C_2^1, I\}$  contains the operation  $S$ , therefore  $\{C_2^1, I\}$  and  $\{C_2^1, S\}$  are identical; see bottom of p. 165.

are evidently distinct from  $C_{2h}^1$ ,  $C_{2h}^2$ ,  $C_{2h}^3$ , respectively), and must therefore contain gliding-reflexions, for they each have a subgroup  $C_2^m$ .

Evidently  $C_s^2$  is the subgroup of  $C_{2h}^4$  and  $C_{2h}^5$ , and  $C_s^4$  that of  $C_{2h}^6$ .

The groups  $C_{2h}^m$  contain an infinite number of centres of symmetry. Their position is at once found by drawing a line  $PQ$  from  $P$  representing *any* translation of the group. Then the group has a centre of symmetry at the middle point of  $PQ$  (p. 155). It may be noted that in the case of  $C_{2h}^1$ ,  $C_{2h}^2$ ,  $C_{2h}^3$  the centres of symmetry form a 'row' of points of interval  $\tau_z$  on *every* axis.

## CHAPTER XIX

## ORTHORHOMBIC GROUPS.

## § 1. ORTHORHOMBIC HEMIMORPHY.

All orthorhombic groups are isomorphous with some orthorhombic point-group; their translation-groups are  $\Gamma_o$ ,  $\Gamma_o'$ ,  $\Gamma_o''$ , or  $\Gamma_o'''$ . Hemimorphic groups are isomorphous with  $C_{2v}$ ; and since  $C_{2v}$  may be derived from  $C_2$  by multiplying by a reflexion in a plane through the axis of  $C_2$ , all groups  $C_{2v}^m$  may be derived by multiplying some group  $C_2^m$  by an operation  $S(t)^*$  which brings the system of axes of  $C_2^m$  into self-coincidence. Such an operation can only be found if the translation-group of  $C_2^m$  has a specialized form, in fact, if  $\Gamma_m$  is specialized to  $\Gamma_o$  or  $\Gamma_o'$ , or  $\Gamma_m$  to  $\Gamma_o'$ ,  $\Gamma_o''$ , or  $\Gamma_o'''$ . Fig. 86 is consequently specialized in one of two ways; either the angle  $GEF$  is a right angle (Fig. 90), or else  $EF = EG$  (Fig. 91).

We may denote as before the axes of  $C_2^m$  which pass through  $E, F, G, H$  by  $e, f, g, h$  respectively. We shall represent by the symbol  $\bullet$  the series of axes obtained by transforming  $e$  by the translations; and  $f, g, h$  will have similar meanings.

If Fig. 86 is so specialized that  $GEF$  is a right angle (but  $EF \neq EG$  in general), the plane of  $S(t)$  must evidently be parallel to one of the sides of the parallelogram  $EFGH$ , and pass through an axis or lie halfway between two axes.

The plane through the axes  $e$  and  $f$  we shall denote by  $\sigma_o$ , that through  $g$  and  $h$  by  $\sigma_o'$ , that through  $e$  and  $g$  by  $\sigma_1$ , that through  $f$  and  $h$  by  $\sigma_1'$ , that through the centre of the parallelogram parallel to the side  $EF$  by  $\sigma_m$ , that through the centre parallel to the side  $EG$  by  $\sigma_{m'}$  (Fig. 90).

If Fig. 86 is so specialized that  $EF = EG$  (but  $GEF$  is not in general a right angle), the plane of  $S(t)$  must evidently

\* Including the case where  $t = \alpha$ . The plane of  $S(t)$  must be parallel to the axes of  $C_2^m$ .

pass through one of the axes and be parallel to a diagonal of the rhombus  $EFGH$ .

We shall denote the plane through  $e$  and  $h$  by  $\sigma_d$ , that through  $f$  and  $g$  by  $\sigma'_d$ , that through  $e$  parallel to  $FG$  by  $\sigma_{d_1}$ , that through  $F$  parallel to  $EH$  by  $\sigma'_d$  (Fig. 91).

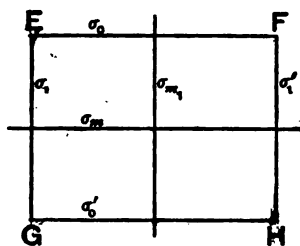


Fig. 90.

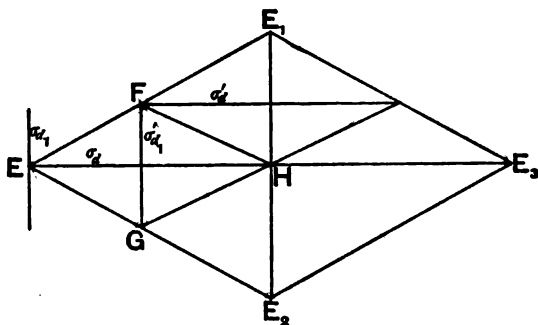


Fig. 91.

$C_{2v}$  has two perpendicular symmetry-planes; therefore all groups  $C_{2v}^m$  have two series of gliding-planes at right angles to one another.

If the axes are arranged as in Fig. 90 these gliding-planes are parallel to  $\sigma_0$  and  $\sigma_1$  respectively; it is evident that we obtain all groups  $C_{2v}^m$  whose axes are so arranged if we always take the plane of  $S(t)$  parallel to one of these, say  $\sigma_0$ .

If the axes are arranged as in Fig. 91 these gliding-planes are parallel to  $\sigma_d$  and  $\sigma'_d$ ; it is evident that we obtain all groups  $C_{2v}^m$  whose axes are so arranged if we always take the plane of  $S(t)$  parallel to one of these, say  $\sigma_d$ .

§ 2. Consider first the groups  $C_{2v}^m$  whose translation-group is  $\Gamma_0$ . Since this is a specialized form of  $\Gamma_m$ , these groups must be derived from groups  $C_2^m$  which have  $\Gamma_m$  as their translation-group, i. e. from  $C_2^1$  or  $C_2^2$ .

The translations of a primitive triplet of  $\Gamma_0$  are mutually orthogonal;  $EE_1 = 2\tau_x$ ,  $EE_2 = 2\tau_y$  (Fig. 86) are perpendicular to one another, and represent a primitive pair in their own plane. The arrangement of axes is therefore that shown in Fig. 90. All the axes of  $C_2^1$  and  $C_2^2$  are geometrically indistinguishable; hence it is sufficient to take the plane of  $S(t)$  as either  $\sigma_0$  or  $\sigma_m$ .

First consider the result of combining  $C_2^1$  with a reflexion  $S_0$  in  $\sigma_0$ . The group so obtained has a subgroup  $C_2^1$  whose

planes are parallel to  $\sigma_0$ ; for  $\Gamma_0$  may be considered as a particular case of a group  $\Gamma_m$  whose 2-al symmetry-axes are parallel to  $EG$ . Since the group  $\{C_2^1, S_0\}$  contains a reflexion in the plane  $\sigma_0$  passing through the 2-al rotation-axis  $e$ , it contains a reflexion in the plane  $\sigma_1$  which is at right angles to  $\sigma_0$  and also passes through  $e$ . The group therefore contains a subgroup  $C_s^1$  whose planes are parallel to  $\sigma_1$ . Hence we have

$$C_{2v}^1 = \{C_2^1, S_0\} = \{C_2^1, S_1\}.$$

Subgroups \*  $C_s^1, C_s^1$ .

This and the following groups are shown diagrammatically in Figs. 92 to 101.

An operation  $A_{\pi, t}$  about  $e$ , combined with a reflexion  $S_0$  in  $\sigma_0$ , is readily seen to be equivalent to an operation  $S_1(t)$  in  $\sigma_1$ . Hence we have

$$C_{2v}^2 = \{C_2^2, S_0\} = \{C_2^2, S_1(\tau_s)\}.$$

Subgroups  $C_s^1, C_s^2$ .

If now the reduced operation in the plane  $\sigma_0$  be not  $S_0$  but  $S_0(t)$  where  $t \neq 0$ ,  $t$  must be  $\tau_x, \tau_x$ , or  $\tau_x + \tau_x$ , for the primitive pair of translations parallel to the plane containing the axes  $e$  and  $f$  is  $2\tau_x, 2\tau_x$ .

Consider, then, the group  $\{C_2^1, S_0(\tau_s)\}$ . The operation  $A_{\pi, 0}$  about  $e$  combined with  $S_0(\tau_s)$  in  $\sigma_0$  is readily seen to be equivalent to an operation  $S_1(\tau_s)$  in  $\sigma_1$ . Hence we have

$$C_{2v}^3 = \{C_2^1, S_0(\tau_s)\} = \{C_2^1, S_1(\tau_s)\}.$$

Subgroups  $C_s^2, C_s^2$ .

$\{C_2^2, S_0(\tau_s)\} = \{C_2^2, S_1\}$  is not geometrically distinguishable from  $C_{2v}^3$ .

Again  $A_{\pi, 0}$  about  $e$  followed by  $S_0(\tau_x)$  in  $\sigma_0$  is equivalent to a reflexion  $S_1$  in  $\sigma_1$  followed by a translation  $\tau_x$ ; i.e. to a reflexion  $S_{m_1}$  in  $\sigma_{m_1}$ . Therefore we have

$$C_{2v}^4 = \{C_2^1, S_0(\tau_x)\} = \{C_2^1, S_{m_1}\}.$$

Subgroups  $C_s^2, C_s^1$ .

Similarly we have

$$C_{2v}^5 = \{C_2^2, S_0(\tau_x)\} = \{C_2^2, S_{m_1}(\tau_s)\}.$$

Subgroups  $C_s^2, C_s^2$ .

\* First is given the subgroup whose planes are parallel to  $\sigma_0$ , then that whose planes are parallel to  $\sigma_1$ .

$$C_{2v}^6 = \{C_2^1, S_0(\tau_x + \tau_s)\} = \{C_2^1, S_{m_1}(\tau_s)\}.$$

Subgroups  $C_s^2, C_s^2$ .

$$C_{2v}^7 = \{C_2^2, S_0(\tau_x + \tau_s)\} = \{C_2^2, S_{m_1}\}.$$

Subgroups  $C_s^2, C_s^1$ .

$$C_{2v}^8 = \{C_2^1, S_m(\tau_x)\} = \{C_2^1, S_{m_1}(\tau_y)\}.$$

Subgroups  $C_s^2, C_s^2$ .

$$C_{2v}^9 = \{C_2^2, S_m(\tau_x)\} = \{C_2^2, S_{m_1}(\tau_y + \tau_s)\}.$$

Subgroups  $C_s^2, C_s^2$ .

$$C_{2v}^{10} = \{C_2^1, S_m(\tau_x + \tau_s)\} = \{C_2^1, S_{m_1}(\tau_y + \tau_s)\}.$$

Subgroups  $C_s^2, C_s^2$ .

There are no more groups  $C_{2v}^m$  whose translation-group is  $\Gamma_0$ , for  $\{C_2^1, S_m\} = \{C_2^1, S_1(\tau_y)\}$  is not geometrically distinguishable from  $C_{2v}^4$ ,  $\{C_2^2, S_m\} = \{C_2^2, S_1(\tau_y + \tau_s)\}$  from  $C_{2v}^7$ ,  $\{C_2^2, S_m(\tau_x + \tau_s)\} = \{C_2^2, S_{m_1}(\tau_y)\}$  from  $C_{2v}^9$ .

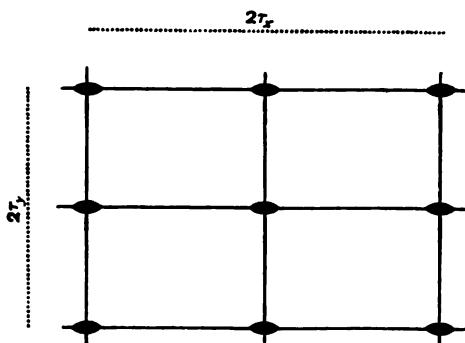


Fig. 92.  $C_{2v}^1$ .

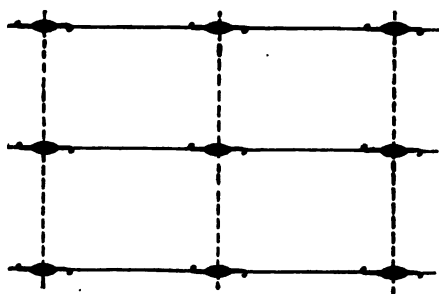


Fig. 93.  $C_{2v}^2$ .

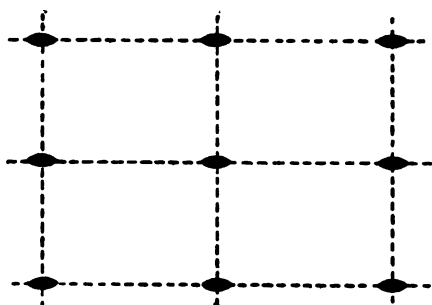


Fig. 94.  $C_{2v}^3$ .

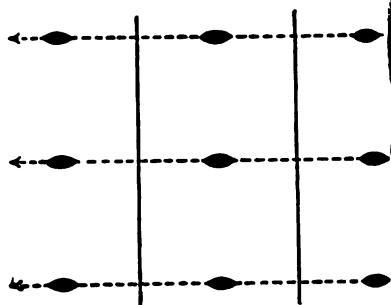


Fig. 95.  $C_{2v}^4$ .

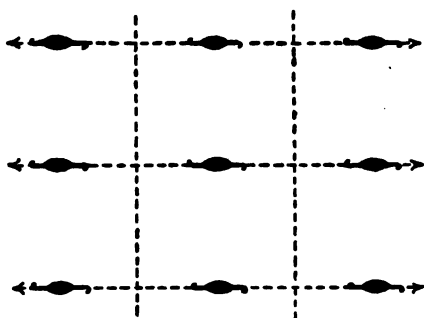


Fig. 96.  $C_{2v}^5$ .

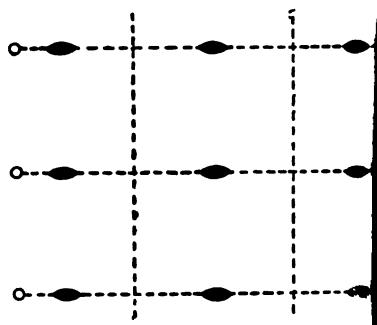


Fig. 97.  $C_{2v}^6$ .

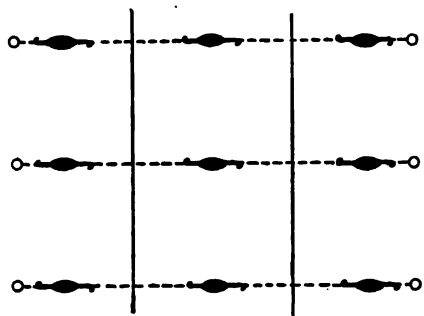


Fig. 98.  $C_{2v}^7$ .

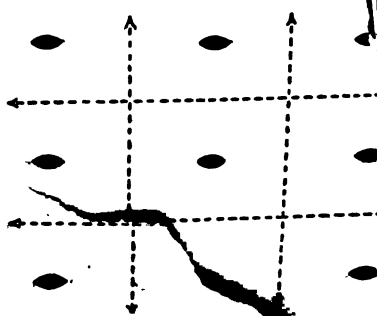


Fig. 99.  $C_{2v}^8$ .

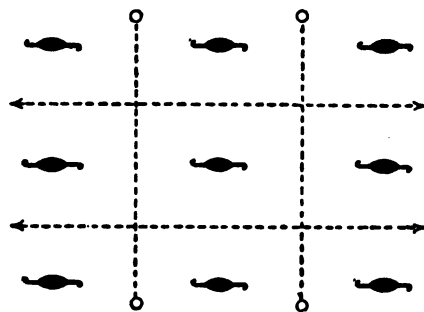


Fig. 100.  $C_{2v}^9$ .

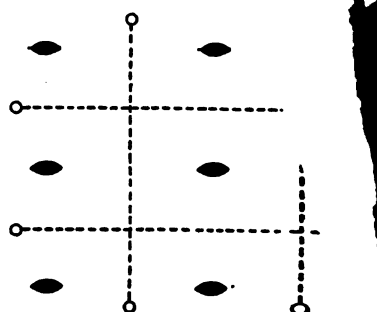


Fig. 101.  $C_{2v}^{10}$ .

§ 3. We shall now find the groups  $C_{2v}^m$  whose translation-group is  $\Gamma_o'$ .

First we shall suppose the axes of that group  $C_2^m$  from which  $C_{2v}^m$  is derived to be in the direction of  $\tau^*$ .  $\Gamma_o'$  is now a specialized form of  $\Gamma_m$  and therefore  $C_2^m$  must be either  $C_1^3$  or  $C_2^3$ . Since a primitive triplet of  $\Gamma_o'$  is  $2\tau_x$ ,  $\tau_x + \tau_y$ ,  $\tau_x - \tau_y$ , the parallelogram  $EE_1E_2E_3$  now takes the specialized form of Fig. 91 in which  $EF = EG$ ;  $EE_3 = 2\tau_x$ ,  $E_1E_2 = 2\tau_y$ .

If such a group  $C_{2v}^m$  contains an operation  $S_d(t)$  in the plane  $\sigma_d$  it must contain a similar operation in a parallel plane through  $E_1$ , since  $EE_1$  is a translation of the group. It contains, however, the operation  $S_d'(t + \tau_x)$  in the plane  $\sigma_d'$ , for this is equivalent to  $S_d(t)$  in  $\sigma_d$  followed by the translation  $\tau_x - \tau_y$ . In fact the subgroups  $C_s^m$  are now either  $C_s^3$  or  $C_s^4$ .

Since no two axes of  $C_2^1$  or of  $C_2^3$  are geometrically distinguishable, it is sufficient to take  $\sigma_d$  as the plane of  $S(t)$ .

We have then (see Figures 102, 103, 104)

$$C_{2v}^{11} = \{C_2^1, S_d\} = \{C_2^1, S_{d_1}\}.$$

Subgroups  $\dagger C_s^2, C_s^3$ .

$$C_{2v}^{12} = \{C_2^3, S_d\} = \{C_2^3, S_{d_1}(\tau_x)\}.$$

Subgroups  $C_s^3, C_s^4$ .

$$C_{2v}^{13} = \{C_2^1, S_d(\tau_x)\} = \{C_2^1, S_{d_1}(\tau_x)\}.$$

Subgroups  $C_s^4, C_s^4$ .

There are no more groups of this sort, for  $\{C_2^3, S_d(\tau_x)\} = \{C_2^3, S_{d_1}\}$  is geometrically equivalent to  $C_{2v}^{12}$ ;

$$\{C_2^1, S_d(\tau_x + \tau_y)\} = \{C_2^1, S_d'(\tau_x)\} \text{ to } C_{2v}^{13}$$

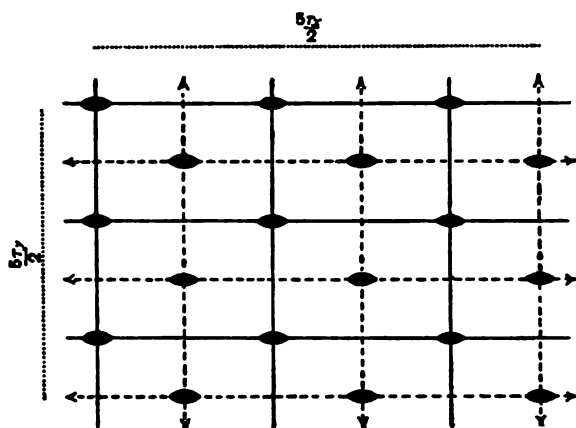
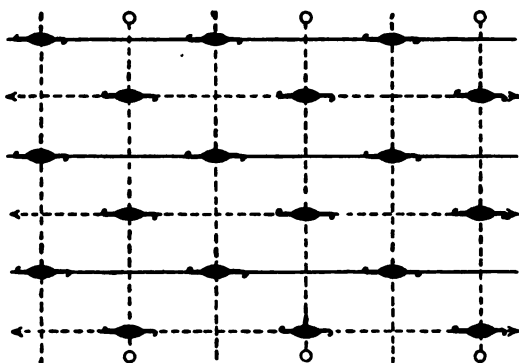
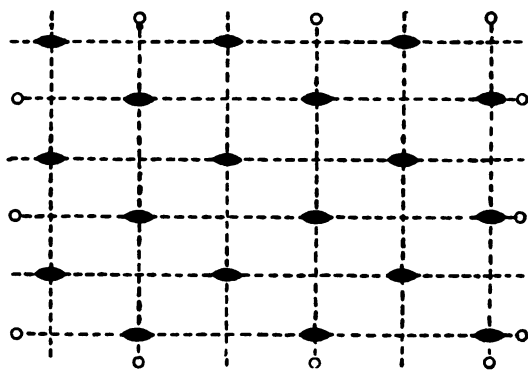
(for the axes  $e, f$  are not geometrically distinguishable);

$$\{C_2^3, S_d(\tau_x + \tau_y)\} = \{C_2^3, S_d'(\tau_x)\} \text{ to } C_{2v}^{12}.$$

Now suppose the axes of  $C_2^m$  to be still in the direction of  $\tau_x$ , but that  $\Gamma_o'$  is not expressed in the notation of p. 186, but is so orientated that  $2\tau_x, \tau_y + \tau_x, \tau_y - \tau_x$  form a primitive triplet, instead of  $2\tau_x, \tau_x + \tau_y, \tau_x - \tau_y$ . In this case  $\Gamma_o'$  considered as a translation-group of  $C_2^m$  is a specialized form of

\* The notation of p. 186 is used.

† First is given the subgroup whose planes are parallel to  $\sigma_d$ , then that whose planes are parallel to  $\sigma_{d_1}$ .

Fig. 102.  $C_{2v}^{11}$ .Fig. 103.  $C_{2v}^{12}$ .Fig. 104.  $C_{2v}^{13}$ .

$\Gamma_m'$  (not of  $\Gamma_m$ ), and therefore  $C_2^m$  must be  $C_2^3$ . (Fig. 105, in which  $EE_1 = 2\tau_x$ ,  $EE_2 = \tau_y$ ,  $E_2R = UE_2 = \tau_z$ ).

The arrangement of axes is evidently that of Fig. 90; the

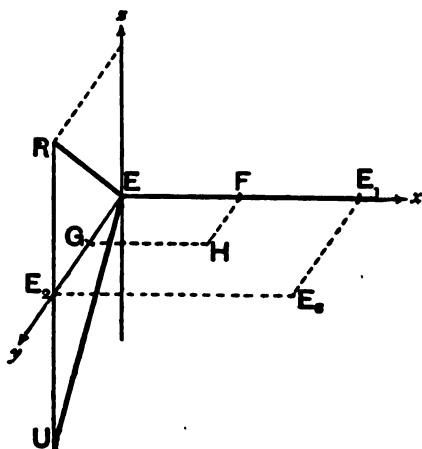


Fig. 105.

series e, f are rotation-axes; g and h are screw-axes.

Now the plane of  $S(t)$  cannot be  $\sigma_m$ , for then this operation would bring screw-axes to coincide with rotation-axes, and therefore would not bring the system of axes of  $C_2^3$  to self-coincidence.

A gliding-reflexion  $S_0(t)$  in  $\sigma_0$  involves a gliding-reflexion  $S_0(t + \tau_z)$  in  $\sigma_0'^*$ ; it is therefore sufficient to take  $\sigma_0$  as the plane of  $S(t)$ .

The subgroups whose planes are parallel to  $\sigma_0$  must be either  $C_2^3$  or  $C_2^1$ ; those whose planes are parallel to  $\sigma_1$  must be either  $C_2^1$  or  $C_2^2$ .

We have (see Figures 106 to 109)

$$C_{2v}^{14} = \{C_2^3, S_0\} = \{C_2^3, S_1\}.$$

Subgroups  $C_2^3, C_2^1$ .

$$C_{2v}^{15} = \{C_2^3, S_0(\tau_z)\} = \{C_2^3, S_1(\tau_z)\}.$$

Subgroups  $C_2^3, C_2^2$ .

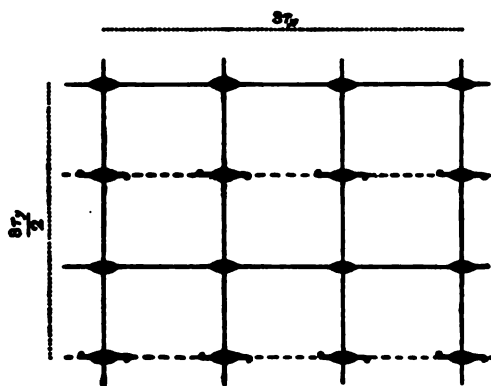
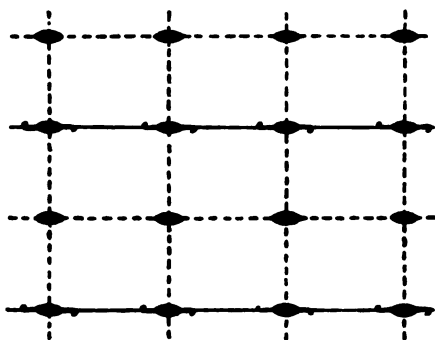
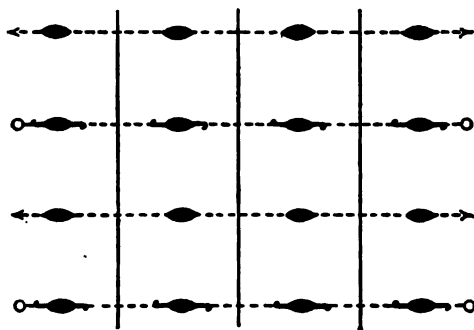
$$C_{2v}^{16} = \{C_2^3, S_0(\tau_x)\} = \{C_2^3, S_{m_1}\}.$$

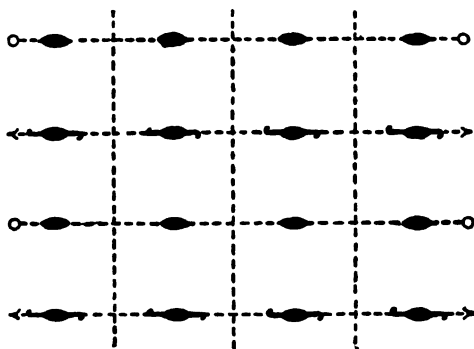
Subgroups  $C_2^4, C_2^1$ .

$$C_{2v}^{17} = \{C_2^3, S_0(\tau_x + \tau_z)\} = \{C_2^3, S_{m_1}(\tau_z)\}.$$

Subgroups  $C_2^4, C_2^2$ .

\* For  $\tau_y + \tau_z$  is a translation of the group.

Fig. 106.  $C_{2v}^{14}$ .Fig. 107.  $C_{2v}^{15}$ .Fig. 108.  $C_{2v}^{16}$ .


 Fig. 109.  $C_{2v}^{17}$ .

§ 4. We now take  $\Gamma_o''$  as translation-group; it is a specialized form of  $\Gamma_m'$ , and therefore the subgroup  $C_2^m$  of  $C_{2v}^m$  must be  $C_2^3$ .

A primitive triplet is  $\tau_y + \tau_z$ ,  $\tau_z + \tau_x$ ,  $\tau_x + \tau_y$ , and hence the arrangement of axes is that of Fig. 90, where  $EF = \frac{1}{2}\tau_x$ ,  $EG = \frac{1}{2}\tau_y$  (compare also Fig. 110 where  $EE' = 2\tau_z$ ). In this case the series of axes  $e$ ,  $h$  are rotation-axes,  $f$ ,  $g$  are screw-axes.

If such a group contains the operation  $S(t)$  in a plane  $\sigma$  parallel to  $\sigma_0$ , it contains an operation of the form  $S(t + \tau_z)$  in another parallel plane distant  $\frac{1}{2}\tau_y$  from  $\sigma$ ; it is only necessary therefore to consider operations  $S(t)$  in  $\sigma_0$  and  $\sigma_m$ .  $t$  must be 0,

$\frac{1}{2}(-\tau_x + \tau_z)$ ,  $\frac{1}{2}(\tau_x + \tau_z)$ , or  $\tau_z$ , for  $\tau_x + \tau_z$ ,  $-\tau_x + \tau_z$  are a primitive pair of translations in the plane perpendicular to  $\tau_y$  (cf. p. 184 (4)). Both subgroups  $C_2^m$  must be either  $C_2^3$  or  $C_2^4$ .

We have then  $C_{2v}^{18} = \{C_2^3, S_0\} = \{C_2^3, S_1\}$ .

Subgroups  $C_2^3$ ,  $C_2^3$ .

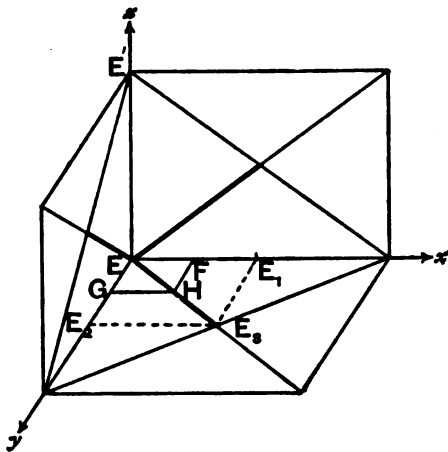


Fig. 110.

$$C_{2v}^{10} = \{C_2^3, S_m(\frac{1}{2}\tau_x + \frac{1}{2}\tau_z)\} = \{C_2^3, S_m(\frac{1}{2}\tau_y + \frac{1}{2}\tau_z)\}.$$

Subgroups  $C_s^4, C_s^4$ .

This exhausts the number of groups whose translation-group is  $\Gamma_o''$ , for  $S_0(-\frac{1}{2}\tau_x + \frac{1}{2}\tau_z)$ ,  $S_0(\frac{1}{2}\tau_x + \frac{1}{2}\tau_z)$ ,  $S_m$ ,  $S_m(\tau_z)$  do not bring the system of axes of  $C_2^3$  into self-coincidence; while  $\{C_2^3, S_0(\tau_z)\} = \{C_2^3, S_0'\}$  is not geometrically distinguishable from  $C_{2v}^{10}$ , for *e* and *h* on the one hand and *f* and *g* on the other cannot be distinguished; and

$$\{C_2^3, S_m(-\frac{1}{2}\tau_x + \frac{1}{2}\tau_z)\} = \{C_2^3, S_m(\frac{1}{2}\tau_x - \frac{1}{2}\tau_z)\} \\ = \{C_2^3, S_m(\frac{1}{2}\tau_x + \frac{1}{2}\tau_z - \tau_z)\}$$

is seen in the same way to be identical with  $C_{2v}^{10}$ .

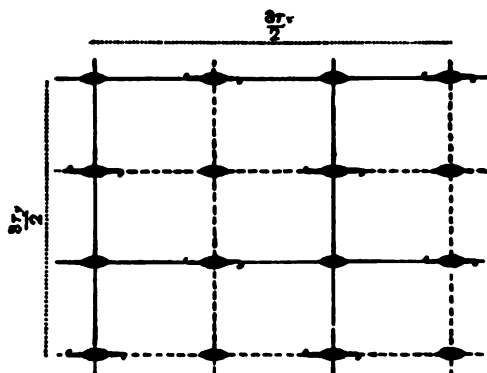


Fig. 111.  $C_{2v}^{10}$ .

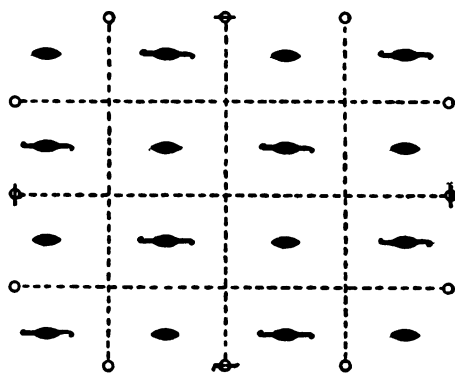


Fig. 112.  $C_{2v}^{10}$ .

In Fig. 111 the planes represented as corresponding to an operation of the form  $S(\tau_x)$ , might be equally well represented as corresponding to  $S(\tau_x)$  for  $\tau_y + \tau_z$  is a translation of the group.

In Fig. 112 the alternate planes in either direction have different translation-components; this is shown by the short lines through the small circles at the ends of the lines representing the alternate planes.

§ 5. Lastly, we consider groups  $C_{2v}^m$  whose translations are

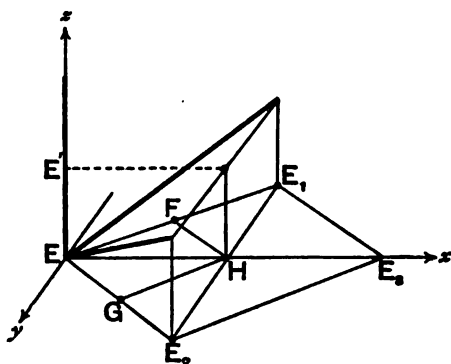


Fig. 113.

represented by  $\Gamma_o'''$ . Since this is a specialized form of  $\Gamma_m'$  the subgroup  $C_2^m$  is  $C_2^3$ .

We take as a primitive triplet of  $\Gamma_o'''$

$$2\tau_x, \tau_x + \tau_y + \tau_z, \tau_x - \tau_y + \tau_z;$$

the arrangement of axes is then that of Fig. 91, where  $EF = \frac{1}{2}(\tau_x - \tau_y)$ ,  $EG = \frac{1}{2}(\tau_x + \tau_y)$ . (See also Fig. 113, where  $EE' = \tau_z$ ). In this case  $e$  and  $h$  are rotation-axes,  $f$  and  $g$  are screw-axes.

If such a group contains an operation  $S(t)$  in a plane  $\sigma$  parallel to  $\sigma_d$ , it contains an operation  $S(t + \tau_x + \tau_z)$  in another parallel plane distant  $\frac{1}{2}\tau_y$  from  $\sigma$ ; it is only necessary therefore to consider operations  $S(t)$  in  $\sigma_d$ .

$t$  must be 0,  $\tau_x$ ,  $\tau_z$ , or  $\tau_x + \tau_z$ , for  $2\tau_x$ ,  $2\tau_z$  are a primitive pair of translations in the plane perpendicular to  $\tau_y$ .\*

Both subgroups  $C_2^m$  must be  $C_2^3$  or  $C_2^4$ .

\* It may be verified from the general condition of p. 124, or otherwise, that the two translations  $2\tau_x$ ,  $(\tau_x + \tau_z + \tau_z) + (\tau_x - \tau_y + \tau_z) - 2\tau_z$  form a primitive pair, if  $\tau_x + \tau_y + \tau_z$ ,  $\tau_x - \tau_y + \tau_z$ ,  $2\tau_z$  are a primitive triplet.

We have

$$C_{2v}^{20} = \{C_2^3, S_d\} = \{C_2^3, S_{d_1}\}.$$

Subgroups  $C_2^3, C_s^3$ .

$$C_{2v}^{21} = \{C_2^3, S_d(\tau_x)\} = \{C_2^3, S_{d_1}(\tau_x)\}.$$

Subgroups  $C_2^4, C_s^4$ .

$$C_{2v}^{22} = \{C_2^3, S_d(\tau_x)\} = \{C_2^3, S_{d_1}'\} = \{C_2^3, S_{d_1}(\tau_y + \tau_x)\}.$$

Subgroups  $C_2^4, C_s^3$ .

$\{C_2^3, S_d(\tau_x + \tau_x)\}$  is not distinguishable from  $\{C_2^3, S_{d_1}(\tau_y + \tau_x)\}$ ,  
i.e. from  $C_{2v}^{22}$ .

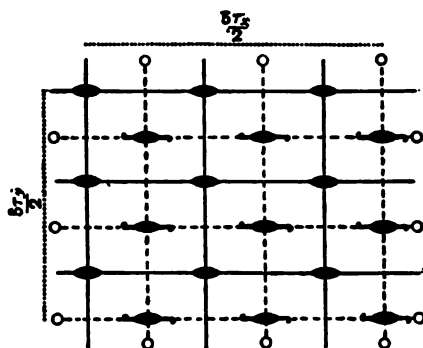


Fig. 114.  $C_{2v}^{20}$ .

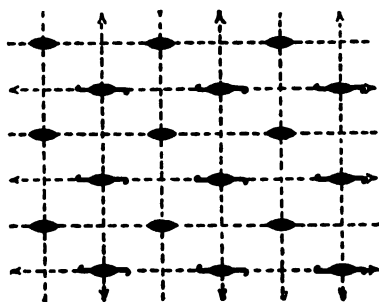


Fig. 115.  $C_{2v}^{21}$ .

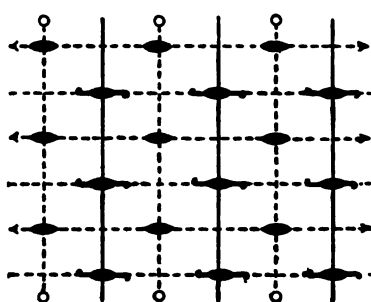


Fig. 116.  $C_{2v}^{22}$ .

## § 6. ORTHORHOMBIC ENANTIOMORPHY.

Since the group  $D_2 (= Q)$  can be derived from  $C_2$  by multiplying the operations of  $C_2$  by a rotation through  $\pi$  about a line perpendicular to the axis of  $C_2$ , therefore all groups

$D_2^m (= Q^m)$  can be derived from  $C_2^m$  by multiplying the operations of  $C_2^m$  by an operation  $A_{\pi, t}$  about an axis  $a$  perpendicular to the axes of  $C_2^m$ ; where  $A_{\pi, t}$  brings the system of axes and translations of  $C_2^m$  into self-coincidence, and is such that  $(A_{\pi, t})^2$  is a translation of  $C_2^m$ . Such an investigation is exactly parallel to that of §§ 2-5, but is rather simpler, as it is only necessary to consider the case of  $t = 0$ , and  $t = \text{half a translation in the direction of } a$ . It will, however, be more convenient for our present purpose to choose another method.

Since  $Q (= D_2)$  has three subgroups  $C_2$  whose axes are mutually orthogonal, therefore  $Q^m (= D_2^m)$  has three subgroups  $C_2^m$  such that the axes of any one of them are at right angles to the axes of both the others. We shall denote these subgroups by  $U, V, W$ ; their axes are parallel to the symmetry-axes of the lattice representing the translation-group (p. 159). We use the notation of pp. 185, 186 and suppose the primitive translations of the lattice in the directions of the axes of  $U, V, W$  to be  $2\tau_x, 2\tau_y, 2\tau_z$  respectively.

If  $a$  is an axis of one of the subgroups  $U, V, W$ , the screw about  $a$  brings the system of axes of each of the other two to self-coincidence; hence evidently  $a$  must meet an axis, or pass halfway between two axes, of each of the other two subgroups.

§ 7. If a group  $Q^m$  has the translation-group  $\Gamma_0$  (primitive triplet  $2\tau_x, 2\tau_y, 2\tau_z$ ), all its subgroups  $C_2^m$  must be either  $C_1^2$  or  $C_2^2$ . There are four possibilities therefore, the subgroups may be  $C_2^1, C_2^1, C_2^1$ ;  $C_2^1, C_2^1, C_2^2$ ;  $C_2^2, C_2^2, C_2^1$ ; or  $C_2^2, C_2^2, C_2^2$ .

If the subgroups  $U, V, W$  are all  $C_2^1$  their axes are all rotation-axes. Any axis of  $U$  must meet an axis of  $V$ ; for were it not so  $W$  must contain screw-axes, since the resultant of two rotations through  $\pi$  about non-intersecting axes is a screw (p. 153(2)), which is contrary to the hypothesis that  $W$  has only rotation-axes.

Through the point of intersection of two rotation-axes a third must pass; hence the arrangement of axes must be that of Fig. 117\*.

$Q^1$ . Subgroups  $C_2^1, C_2^1, C_2^1$ .

Again, if two of the subgroups, e.g.  $U$  and  $V$ , are  $C_2^1$ , but the third is  $C_2^2$ ; the axes of  $U, V$  are all rotation-axes and those of  $W$  are all screw-axes. No axis of  $U$  can meet an

\* The explanation of these figures is given on p. 190.

axis of  $V$ , but the line perpendicular to and meeting a rotation-axis of  $U$  and a rotation-axis of  $V$  must be a screw-axis of  $W$ . Any rotation-axis of the group meets screw-axes and passes halfway between rotation-axes whose direction is perpendicular to it (Fig. 118).

$Q^2$ . Subgroups  $C_2^1, C_2^1, C_2^2$ .

If  $U$  and  $V$  are  $C_2^2$  and  $W$  is  $C_2^1$  the axes of  $U$  and  $V$  are all screw-axes, those of  $W$  all rotation-axes. No axis of  $W$  can cut an axis of  $U$  or  $V$ , for else the group would have rotation-axes whose direction is that of the axes of  $V$  or  $U$  respectively (p. 154(4)). On the other hand the axes of  $U$  and  $V$  cut, for otherwise  $W$  would contain screw-axes. Therefore the arrangement of axes must be that of Fig. 119.

$Q^3$ . Subgroups  $C_2^2, C_2^2, C_2^1$ .

If  $U, V, W$  are all  $C_2^2$ , all the axes of the group are screw-axes and therefore (p. 153(3)) no two axes cut. This leads of necessity to the arrangement of Fig. 120.

$Q^4$ . Subgroups  $C_2^2, C_2^2, C_2^2$ .

In Figs. 117 to 125 the arrangement of the axes of the groups  $Q^1$  to  $Q^9$  is shown.

Rotation-axes are denoted by thick lines, screw-axes by thick broken lines, and the sides of a rectangular parallelepiped of sides  $\tau_x, \tau_y, \tau_z$  by ordinary lines\*. The complete series of axes of any one of these groups would be given by filling space with repetitions of the corresponding figure, arranged in the same way as the parallelepipeda of Fig. 66.

§ 8. If the translation-group is  $\Gamma_o'$  (primitive triplet  $\tau_x + \tau_y, \tau_x - \tau_y, 2\tau_z$ )  $U$  and  $V$  must be  $C_2^2$ , and  $W$  is either  $C_2^1$  or  $C_2^2$  (cp. § 3, pp. 181 to 183).

The arrangement of the axes of  $W$  is that of Fig. 91; and since the operations of  $U$  and  $V$  bring this system of axes into self-coincidence, the axes of  $U$  and  $V$  must be parallel to the diagonals of that figure.

In the group whose subgroups are  $C_2^2, C_2^2, C_2^2$  the axes of  $W$  are all screw-axes, and therefore no two rotation-axes of  $U$  and  $V$  can meet. Since, however, a rotation about one of the rotation-axes of  $U$ , followed by a rotation about any one of the rotation-axes of  $V$ , is equivalent to a screw about a line meeting both axes, therefore the screw-axes of  $W$  cut

\* Except in the case of Fig. 123, when the sides of the parallelepiped are  $\frac{1}{2}\tau_x, \frac{1}{2}\tau_y, \frac{1}{2}\tau_z$ .

rotation-axes of  $U$  and  $V$ . These considerations show that the arrangement of axes must be that of Fig. 121.

$Q^6$ . Subgroups  $C_2^3, C_2^3, C_2^3$ .

If however the axes of  $W$  are rotation-axes, the rotation-axes of  $U$  and  $V$  must intersect, and axes of  $W$  must pass through their points of intersection; this leads at once to Fig. 122.

$Q^6$ . Subgroups  $C_2^3, C_2^3, C_2^1$ .

§ 9. If the translation-group is  $\Gamma_o''$  (primitive triplet  $\tau_y + \tau_z, \tau_z + \tau_x, \tau_x + \tau_y$ )  $U, V, W$  are all  $C_2^3$ .

The arrangement of the axes of  $U, V$ , and  $W$  is that shown in Fig. 90, where  $e, h$  are rotation-axes and  $f, g$  screw-axes (p. 185).

Since a rotation through  $\pi$  about any rotation-axis of  $U$  brings the system of axes of  $W$  into self-coincidence, therefore any rotation-axis of  $U$  must meet rotation-axes and screw-axes of  $W$ . Through the points where it meets the former, rotation-axes of  $V$  must pass; through the points where it meets the latter axes of  $V$  pass which cannot be rotation-axes, and which are therefore screw-axes. These considerations with the help of Fig. 90 lead at once to the arrangement of Fig. 123.

$Q^7$ . Subgroups  $C_2^3, C_2^3, C_2^3$ .

In Fig. 123 the sides of the parallelepiped are  $\frac{1}{2}\tau_x, \frac{1}{2}\tau_y, \frac{1}{2}\tau_z$  not  $\tau_x, \tau_y, \tau_z$  as in the other cases.

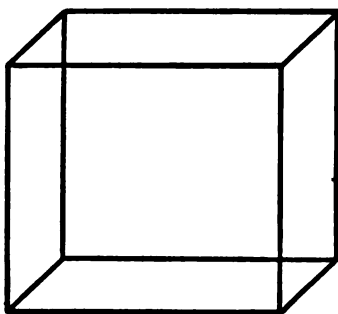
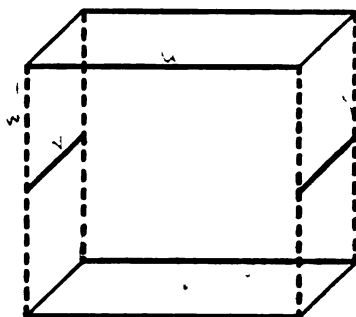
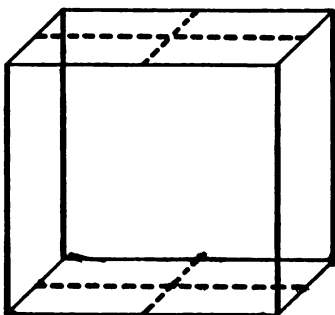
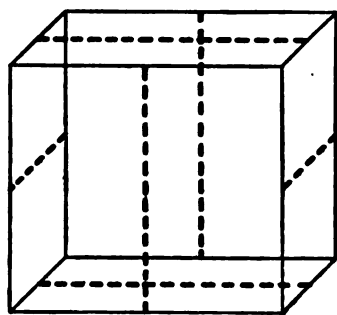
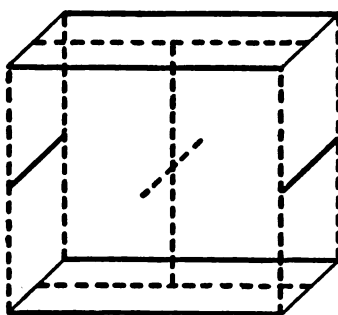
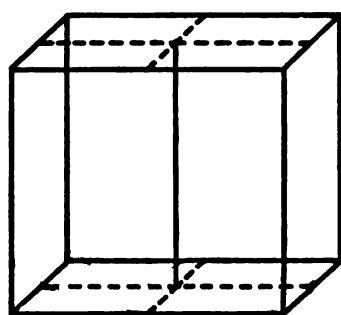
§ 10. If the translation-group is  $\Gamma_o'''$  (for which  $\tau_x + \tau_y + \tau_z$ , and any two of  $2\tau_x, 2\tau_y, 2\tau_z$  form a primitive triplet)  $U, V, W$  are all  $C_2^3$ . The arrangement of the axes of  $U, V$ , and  $W$  is that of Fig. 91, where  $e, h$  are rotation-axes and  $f, g$  screw-axes (p. 187). A rotation-axis of  $U$  must pass either through two rotation-axes or two screw-axes of  $W$ \*. In the former case rotation-axes of  $V$  pass through the intersection of rotation-axes of  $U$  and  $W$ , which consideration with the help of Fig. 91 leads at once to Fig. 124.

$Q^8$ . Subgroups  $C_2^3, C_2^3, C_2^3$ .

In the latter case a rotation-axis of  $U$  cannot meet a rotation-axis of  $V$ ; but it meets screw-axes of  $V$ , which intersect and are perpendicular to rotation-axes of  $U$  and  $W$ . The arrangement of axes is easily seen to be that of Fig. 125.

\* This is evident from an inspection of Fig. 91, since the rotation about this axis of  $U$  must bring the system of axes of  $W$  into self-coincidence.

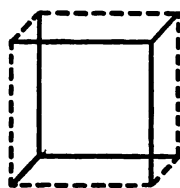
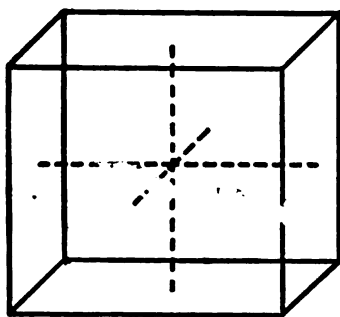
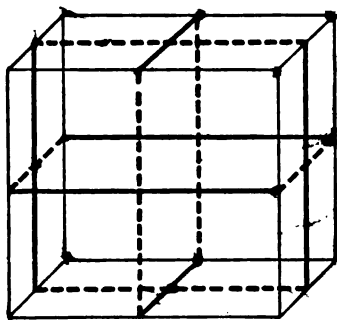
§ 11. The reader will find it a great assistance to the understanding of what follows if he will make himself thoroughly familiar with Figs. 117 to 125.

Fig. 117.  $Q^1$ .Fig. 118.  $Q^2$ .Fig. 119.  $Q^3$ .Fig. 120.  $Q^4$ .Fig. 121.  $Q^5$ .Fig. 122.  $Q^6$ .

The above method has the advantage that it shows most

readily the fact that no group  $Q^m$  can have its axes arranged in any way other than those shown in Figs. 117 to 125. It is not quite so clear perhaps that to each of these arrangements a group really corresponds. This may be shown as follows.

Let  $A$  be the operation of  $U$  about any one ( $a$ ) of the axes of  $U$  as shown in any one of these figures. Then in each case  $A$  brings the system of axes of  $W$  to self-coincidence, and  $A^2$  is a translation of the group. Therefore, by p. 164,  $\{W, A\}$  is really a group; and in each figure the


 Fig. 123.  $Q^7$ .

 Fig. 124.  $Q^8$ .

 Fig. 125.  $Q^9$ .

axes of  $U$  other than  $a$  and the axes of  $V$  are a necessary consequence of the coexistence of  $a$  and the axes of  $W$ .

## § 12. ORTHORHOMBIC HOLOHEDRY.

Since the point-group  $Q_h$  is obtained by multiplying the operations of  $Q$  by an inversion, therefore the groups  $Q_h^m$  are derived from such groups  $Q^m$  as satisfy the condition of p. 165, that is, from *all* groups  $Q^m$ , by multiplying by an inversion which brings the system of axes of  $Q^m$  into self-coincidence. Since an inversion about a point  $P$  followed by an inversion about a point  $P'$  is equivalent to a translation  $2PP'$ , therefore two groups  $Q_h^m$  formed from the same group  $Q^m$  by multiplying by inversions about the points  $P, P'$  are only identical if  $PP'$  is half a translation of the group, or if  $P$  and  $P'$  are similarly situated with regard to the axes of  $Q^m$ . Inversion about a centre of symmetry of  $Q^m$  must bring the system of axes of  $Q^m$  into self-coincidence, and therefore every centre of symmetry must lie in a vertex,

the middle, the middle of an edge, or the middle of a face of a rectangular parallelepiped  $p$ , such as is shown in Figs. 117 to 125.

We shall denote the centre of  $p$  by  $R$ , the middle point of its edges by  $L, M, N$ , &c.; the middle points of its faces by  $D, E, F$ , &c.; and its vertices by  $A, A_1$ , &c. (Fig. 126).

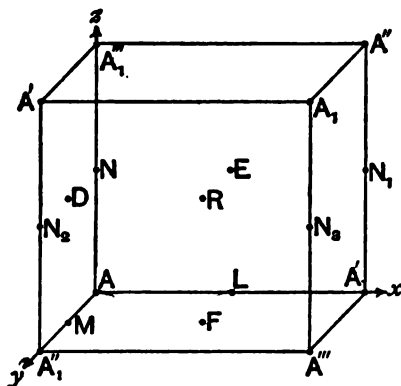


Fig. 126.

Unless  $p$  is the parallelepiped of Fig. 123, a group which has a centre of symmetry at one of the vertices of  $p$  has centres of symmetry at all the other vertices; for each edge of  $p$  represents half a translation of the group.

Since  $Q_h$  has three subgroups  $C_{2h}$  and three subgroups  $C_{2v}$ ,  $Q_h^m$  has three subgroups  $C_{2h}^m$  and three subgroups  $C_{2v}^m$ . These will

be given for each group; that subgroup which has its axes in the direction of  $\tau_x$  being put first, then that which has its axes in the direction of  $\tau_y$ , and lastly, that which has its axes in the direction of  $\tau_z$ . A reference to the figures given for the groups  $Q^m$  and  $C_{2v}^m$  will be sufficient to show completely the arrangement of all the axes and planes of each group  $Q_h^m$ . Every plane of symmetry or gliding-plane is perpendicular to one of the three series of axes and parallel to the other two.

§ 13. First consider the groups whose translation-group is  $\Gamma_o$ , of which  $2\tau_x, 2\tau_y, 2\tau_z$  form a primitive triplet.

Starting from  $Q^1$  we have \*

$$Q_h^1 = \{Q^1, I\}.$$

$$\text{Subgroups } C_{2h}^1, C_{2h}^1, C_{2h}^1; C_{2v}^1, C_{2v}^1, C_{2v}^1.$$

$$Q_h^2 = \{Q^1, I_r\}.$$

$$\text{Subgroups } C_{2h}^2, C_{2h}^2, C_{2h}^2; C_{2v}^{10}, C_{2v}^{10}, C_{2v}^{10}.$$

$L, M, N$  have similar situations with regard to the axes of  $Q^1$ , the same is true of  $D, E, F$ ; hence the only other groups

\* Inversions about  $A, R, L, M, N, D, E, F$  are denoted by  $I, I_r, I_l, I_m, I_n, I_d, I_e, I_f$  respectively.

$Q_h^m$  derivable from  $Q^1$  are

$$Q_h^3 = \{Q^1, I_n\}.$$

Subgroups  $C_{2h}^4, C_{2h}^4, C_{2h}^7; C_{2v}^4, C_{2v}^4, C_{2v}^8.$

$$Q_h^4 = \{Q^1, I_f\}.$$

Subgroups  $C_{2h}^4, C_{2h}^4, C_{2h}^4; C_{2v}^6, C_{2v}^6, C_{2v}^8.$

$R$  and  $F$  have similar situations with regard to the axes of  $Q^2$ , and so have  $A$  and  $N, L$  and  $D, M$  and  $E$ ; hence the only distinct groups derivable from  $Q^2$  are

$$Q_h^5 = \{Q^2, I\}.$$

Subgroups  $C_{2h}^7, C_{2h}^4, C_{2h}^2; C_{2v}^4, C_{2v}^7, C_{2v}^8.$

$$Q_h^6 = \{Q^2, I_r\}.$$

Subgroups  $C_{2h}^4, C_{2h}^4, C_{2h}^5; C_{2v}^6, C_{2v}^{10}, C_{2v}^8.$

$$Q_h^7 = \{Q^2, I_l\}.$$

Subgroups  $C_{2h}^7, C_{2h}^4, C_{2h}^5; C_{2v}^6, C_{2v}^4, C_{2v}^7.$

$$Q_h^8 = \{Q^2, I_m\}.$$

Subgroups  $C_{2h}^4, C_{2h}^4, C_{2h}^5; C_{2v}^8, C_{2v}^8, C_{2v}^6.$

$L$  and  $M, D$  and  $E$  have similar situations with regard to the axes of  $Q^3$ ; the distinct groups derivable from  $Q^3$  are

$$Q_h^9 = \{Q^3, I\}.$$

Subgroups  $C_{2h}^6, C_{2h}^8, C_{2h}^1; C_{2v}^2, C_{2v}^2, C_{2v}^8.$

$$Q_h^{10} = \{Q^3, I_r\}.$$

Subgroups  $C_{2h}^6, C_{2h}^8, C_{2h}^4; C_{2v}^9, C_{2v}^9, C_{2v}^8.$

$$Q_h^{11} = \{Q^3, I_m\}.$$

Subgroups  $C_{2h}^9, C_{2h}^8, C_{2h}^4; C_{2v}^5, C_{2v}^9, C_{2v}^4.$

$$Q_h^{12} = \{Q^3, I_n\}.$$

Subgroups  $C_{2h}^8, C_{2h}^8, C_{2h}^1; C_{2v}^7, C_{2v}^7, C_{2v}^{10}.$

$$Q_h^{13} = \{Q^3, I_f\}.$$

Subgroups  $C_{2h}^9, C_{2h}^8, C_{2h}^4; C_{2v}^7, C_{2v}^7, C_{2v}^7.$

$$Q_h^{14} = \{Q^3, I_d\}.$$

Subgroups  $C_{2h}^6, C_{2h}^8, C_{2h}^4; C_{2v}^9, C_{2v}^6, C_{2v}^6.$

$A$  and  $R$  have similar situations with regard to the axes of  $Q^4$  and so have  $L, M, N, D, E, F$ ; the distinct groups derivable from  $Q^4$  are therefore

$$Q_h^{15} = \{Q^4, I\}.$$

Subgroups  $C_{2h}^6, C_{2h}^8, C_{2h}^8; C_{2v}^6, C_{2v}^6, C_{2v}^6.$

$$Q_h^{16} = \{Q^4, I_e\}.$$

Subgroups  $C_{2h}^6, C_{2h}^8, C_{2h}^8; C_{2v}^7, C_{2v}^7, C_{2v}^6.$

§ 14. Now let the translation-group be  $\Gamma_o'$ , of which  $2\tau_x$ ,  $\tau_x + \tau_y$ ,  $\tau_x - \tau_y$  is a primitive triplet.  $AF$ ,  $NR$ ,  $LM$ ,  $DE$  are now half translations, so that if a group  $Q_h^m$  whose translation-group is  $\Gamma_o'$  has a centre of symmetry at  $A$  it has one at  $F$ , and if it has a centre at  $R$  it has one at  $N$ , &c.

Again,  $A$  and  $N$ ,  $L$  and  $D$  are similarly situated with regard to the axes of  $Q^5$ , but not with regard to the axes of  $Q^6$ ; therefore the distinct groups derivable from  $Q^5$  are

$$Q_h^{17} = \{Q^5, I\}.$$

$$\text{Subgroups } C_{2h}^3, C_{2h}^6, C_{2h}^3; C_{2v}^{16}, C_{2v}^{14}, C_{2v}^{12}.$$

$$Q_h^{18} = \{Q^5, I_l\}.$$

$$\text{Subgroups } C_{2h}^3, C_{2h}^6, C_{2h}^3; C_{2v}^{17}, C_{2v}^{15}, C_{2v}^{13}.$$

And those derivable from  $Q^8$  are

$$Q_h^{19} = \{Q^6, I\}.$$

$$\text{Subgroups } C_{2h}^3, C_{2h}^3, C_{2h}^3; C_{2v}^{14}, C_{2v}^{14}, C_{2v}^{11}.$$

$$Q_h^{20} = \{Q^6, I_r\}.$$

$$\text{Subgroups } C_{2h}^3, C_{2h}^6, C_{2h}^3; C_{2v}^{16}, C_{2v}^{16}, C_{2v}^{13}.$$

$$Q_h^{21} = \{Q^6, I_l\}.$$

$$\text{Subgroups } C_{2h}^3, C_{2h}^3, C_{2h}^3; C_{2v}^{15}, C_{2v}^{15}, C_{2v}^{11}.$$

$$Q_h^{22} = \{Q^6, I_d\}.$$

$$\text{Subgroups } C_{2h}^3, C_{2h}^6, C_{2h}^3; C_{2v}^{17}, C_{2v}^{17}, C_{2v}^{13}.$$

§ 15. Now let the translation-group be  $\Gamma_o'''$  of which a primitive triplet is  $\tau_y + \tau_z$ ,  $\tau_z + \tau_x$ ,  $\tau_x + \tau_y$ . If  $A$  is a centre of symmetry,  $A'$ ,  $A''$ ,  $A'''$  are also centres of symmetry, but  $A_1$ ,  $A_1'$ ,  $A_1''$ ,  $A_1'''$  are not\*.

However the group formed by combining an inversion about  $A$  with the operations of  $Q^7$  (which is the only group  $Q^m$  whose translation-group is  $\Gamma_o'''$ ) is the same as that formed by combining an inversion about  $A_1$ , for  $A$  and  $A_1$  have similar situations with regard to the axes of  $Q^7$ . Now the only other kind of inversion which brings the system of axes of  $Q^7$  into self-coincidence is that about  $R$  (Fig. 126); hence the only two groups of this sort are

$$Q_h^{23} = \{Q^7, I\}.$$

$$\text{Subgroups } C_{2h}^3, C_{2h}^3, C_{2h}^3; C_{2v}^{18}, C_{2v}^{18}, C_{2v}^{18}.$$

$$Q_h^{24} = \{Q^7, I_r\}.$$

$$\text{Subgroups } C_{2h}^3, C_{2h}^6, C_{2h}^3; C_{2v}^{19}, C_{2v}^{19}, C_{2v}^{19}.$$

\* Note that the sides of the parallelepiped of Fig. 123 are only  $\frac{1}{2}\tau_x$ ,  $\frac{1}{2}\tau_y$ ,  $\frac{1}{2}\tau_z$ .

§ 16. Lastly, we consider the translation-group  $\Gamma_o'''$ . This contains the translation  $\tau_x + \tau_y + \tau_z$ , so that a group possessing an inversion about  $A$  has one about  $R$ ; similarly a group with an inversion about  $L$ ,  $M$ , or  $N$  has one about  $D$ ,  $E$ , or  $F$  respectively. Again  $L$ ,  $M$ ,  $N$  have similar situations with regard to the axes both of  $Q^8$  and  $Q^9$ ; hence we have only

$$Q_h^{25} = \{Q^8, I\}.$$

$$\text{Subgroups } C_{2h}^8, C_{2h}^8, C_{2h}^8; C_{2v}^{20}, C_{2v}^{20}, C_{2v}^{20}.$$

$$Q_h^{26} = \{Q^8, I_n\}.$$

$$\text{Subgroups } C_{2h}^8, C_{2h}^8, C_{2h}^8; C_{2v}^{22}, C_{2v}^{22}, C_{2v}^{21}.$$

$$Q_h^{27} = \{Q^9, I\}.$$

$$\text{Subgroups } C_{2h}^9, C_{2h}^9, C_{2h}^9; C_{2v}^{21}, C_{2v}^{21}, C_{2v}^{21}.$$

$$Q_h^{28} = \{Q^9, I_n\}.$$

$$\text{Subgroups } C_{2h}^9, C_{2h}^9, C_{2h}^9; C_{2v}^{22}, C_{2v}^{22}, C_{2v}^{20}.$$

## CHAPTER XX

## TETRAGONAL GROUPS.

§ 1. The translation-group of any one of these groups is either  $\Gamma_t$  (primitive triplet,  $2\tau_x, 2\tau_y, 2\tau_z$ ) or  $\Gamma'_t$  (primitive triplet,  $\tau_y + \tau_z, \tau_z + \tau_x, \tau_x + \tau_y$ ).

In either case  $\tau_x$  and  $\tau_y$  are equal in magnitude, and the 4-al axes of the groups are in the direction of  $\tau_x$ .

Any plane perpendicular to  $\tau_x$  will be called a 'principal' plane.

## § 2. TETRAGONAL TETARTOEDRY OF THE SECOND SORT.

The point-group  $C_4'$  is obtained by multiplying the operations of  $C_2$  by a rotatory-reflexion of angle  $\frac{\pi}{2}$ . Hence all space-groups  $C_4'^m$  are obtained by combining the operations of a group  $C_2^m$  with an operation  $A'$ , which must bring the system of axes of  $C_2^m$  to self-coincidence, be isomorphous with a rotatory-reflexion of angle  $\frac{\pi}{2}$ , and be such that  $A'^2$  is an operation of  $C_2^m$  (p. 164). These conditions are satisfied if and only if  $A'$  is a rotatory-reflexion of angle  $\frac{\pi}{2}$  (see p. 154), whose axis coincides with a rotation-axis of  $C_2^m$ .  $C_2^2$  has no rotation-axis; therefore we have evidently the two groups

$$C_4'^1 = \{C_2^1, A'\}$$

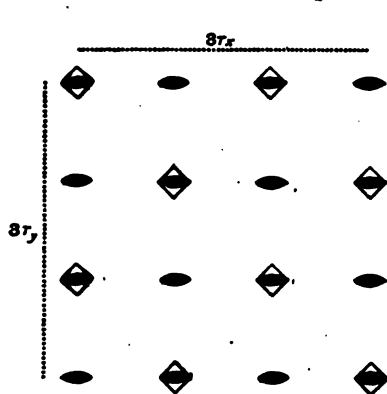
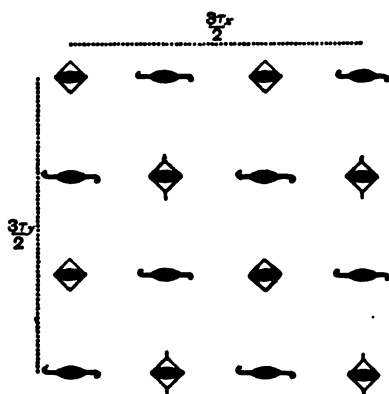
$$C_4'^2 = \{C_2^2, A'\}$$

of this class, and these two only.

The arrangement of their axes is shown in Figs. 127 and 128, in which  $\blacklozenge$  denotes one of a series of 4-al axes whose planes coincide, and  $\blacklozenge$  one of a series whose planes are at a distance  $\frac{\tau_x}{2}$  from those of the other series.

Since  $A'$  followed by the translation  $2m\tau_x$  is equivalent to a rotatory-reflexion, whose axis is the axis of  $A'$  and whose plane is at a distance  $m\tau_x$  from the plane of  $A'$ , these groups

contain an indefinite number of rotatory-reflexions about the same axis. The corresponding planes of rotatory-reflexion form a set of interval  $\tau_z$ .

Fig. 127.  $C_4^1$ .Fig. 128.  $C_4^3$ .

### § 8. TETRAGONAL HEMIHEDRY OF THE SECOND SORT.

The group  $D_{2d}$  is derived from  $D_2$  by multiplying by a reflexion in a plane parallel to one axis ( $w$ ) and bisecting the angle between the other two; therefore groups  $D_{2d}^m$  are derived from  $D_2^m$  (or  $Q^m$ ) by combining with an operation  $S(t)$  in a plane parallel to one set of axes and bisecting the angle between the other two sets which brings the system of axes to self-coincidence. In order that this last condition may be fulfilled the plane of  $S(t)$  must evidently be parallel to one of the planes  $NRFA$ ,  $DELM$  of Fig. 126\*; we shall take it parallel to  $NRFA$ †.

Again, since the axis  $w$  is a 4-al axis of the second sort for  $D_{2d}$ , therefore  $D_{2d}^m$  contains operations isomorphous with rotatory-reflexions of angle  $\frac{\pi}{2}$ ; that is,  $D_{2d}^m$  has 4-al axes of the second sort. These must be 2-al rotation-axes of the group  $Q^m$  from which  $D_{2d}^m$  is derived; hence no groups  $D_{2d}^m$  can be derived from  $Q^2$ ,  $Q^4$ , and  $Q^6$  which have no rotation-axes parallel to  $AN$  (Fig. 126). The position of the 4-al axes of the second sort may be readily found in each case. Suppose

\* This is so in the case of  $Q^2$ ,  $Q^4$ ,  $Q^6$  and  $Q^8$ ; in the other groups  $Q^m$  we may still take the plane of  $S(t)$  parallel to  $NRFA$ , since the arrangement of all three sets of axes is the same.

† Of course, if a group  $D_{2d}^m$  has an operation  $S(t)$  in a plane parallel to  $DELM$  it has a similar operation in a plane parallel to  $NRFA$ .

that we have a group containing the screw  $A_{\pi, t_1}$  about an axis  $a$ , and the operation  $S(t)$  in a plane  $\sigma$  making an angle  $\frac{\pi}{4}$  with  $a$ . Then  $A_{\pi, t_1} \cdot S(t) =$  the translation  $t_1$ , followed by  $A_{\pi, 0} \cdot S$  and then by the translation  $t =$  the translation  $t_1$ , followed by a rotatory-reflexion of angle  $\frac{\pi}{2}$  about an axis perpendicular to  $a$ , lying in  $\sigma$ , and passing through the intersection of  $a$  and  $\sigma$  (p. 68), and then by the translation  $t$ . This is equivalent to a rotatory-reflexion of angle  $\frac{\pi}{2}$  about an axis easily found (p. 154).

Of course the translation-groups of  $Q^m$  must be specialized,  $\Gamma_0$  and  $\Gamma_0'$  to  $\Gamma_t$ , and  $\Gamma_0''$  and  $\Gamma_0'''$  to  $\Gamma_t'$ .

In groups  $D_{2d}^m$ , derived from  $Q^1$ ,  $Q^3$ , and  $Q^7$ ,  $\tau_x$  and  $\tau_y$  are parallel to  $AL$  and  $AM$  of Fig. 126; in groups derived from  $Q^6$ ,  $Q^8$ , and  $Q^9$ ,  $\tau_x$  and  $\tau_y$  are parallel to  $ML$  and  $AF$ . This will be clear from a consideration of pp. 137 and 138.

The arrangement of all the axes and planes of any group  $D_{2d}^m$  is given when we know the subgroups  $Q^m$  and  $C_{2v}^m$ .

Again, two groups derived from the same group  $Q^m$  and having the same subgroup  $C_{2v}^m$  are identical. For if  $\{Q^m, S(t)\}$  and  $\{Q^m, S'(t')\}$  have the same subgroup  $C_{2v}^m$ ,  $C_{2v}^m$  must contain both  $S(t)$  and  $S'(t')$ . Therefore, since  $\{Q^m, S(t)\}$  has  $C_{2v}^m$  as a subgroup and  $C_{2v}^m$  contains  $S'(t')$ ,  $\{Q^m, S(t)\}$  contains  $S'(t')$ .

Hence  $\{Q^m, S(t)\}$  and  $\{Q^m, S'(t')\}$  are identical (p. 165).

It follows from the translations in every case that if the plane  $NRFA$  is a gliding-plane of any group  $D_{2d}^m$ , so are the parallel planes through  $A_1'$  and  $A_1''$ ; and if a plane through  $L$  parallel to  $NRFA$  is a gliding-plane, so is the parallel plane through  $M$ . Hence we need only consider operations  $S(t)$  in the plane  $NRFA$  ( $\sigma_d$ ) and in the parallel plane through  $M$  ( $\sigma_{d_1}$ ). The corresponding operations will be denoted by  $S_d(t)$  and  $S_{d_1}(t)$  respectively.

§ 4. First suppose the translation-group to be  $\Gamma_t$ .

We have

$$D_{2d}^1 = \{Q^1, S_d\} = \{Q^1, S_d(\tau_x + \tau_y)\}^*.$$

Subgroup  $C_{2v}^{11}$ ; 4-al axes  $A, A'''$  †.

\*  $\{Q^1, S_d\}$  and  $\{Q^1, S_d(\tau_x + \tau_y)\}$  are identical because they have the same subgroup  $C_{2v}^{11}$ ; the reasoning is the same in similar cases.

† These are the points where the axis and a plane of a 4-al axis of the second sort meet.

$$D_{2d}^2 = \{Q^1, S_d(\tau_z)\} = \{Q^1, S_d(\tau_x + \tau_y + \tau_z)\}.$$

Subgroup  $C_{2v}^{13}$ ; 4-al axes  $N, N_3$ .

$$D_{2d}^3 = \{Q^3, S_d\} = \{Q^3, S_d(\tau_x + \tau_y)\}.$$

Subgroup  $C_{2v}^{11}$ ; 4-al axes  $A_1', A_1''$ .

$$D_{2d}^4 = \{Q^3, S_d(\tau_z)\} = \{Q^3, S_d(\tau_x + \tau_y + \tau_z)\}.$$

Subgroup  $C_{2v}^{13}$ ; 4-al axes  $N_1, N_3$ .

No operation in the plane  $\sigma_{d_1}$  brings the system of axes of  $Q^1$  or  $Q^3$  to self-coincidence.

Similarly, we have from  $Q^6$

$$D_{2d}^5 = \{Q^6, S_d\}.$$

Subgroup  $C_{2v}^{13}$ ; 4-al axes  $A, A_1', A_1'', A'''$ .

$$D_{2d}^6 = \{Q^6, S_d(\tau_z)\}.$$

Subgroup  $C_{2v}^{13}$ ; 4-al axes  $N, N_1, N_2, N_3$ .

$$D_{2d}^7 = \left\{Q^6, S_{d_1}\left(\frac{\tau_x + \tau_y}{2}\right)\right\}.$$

Subgroup  $C_{2v}^{13}$ ; 4-al axes  $F$ .

$$D_{2d}^8 = \left\{Q^6, S_{d_1}\left(\frac{\tau_x + \tau_y}{2} + \tau_z\right)\right\}.$$

Subgroup  $C_{2v}^{10}$ ; 4-al axes  $R$ .

The operations  $S_{d_1}, S_{d_1}(\tau_z), S_d\left(\frac{\tau_x + \tau_y}{2}\right), S_d\left(\frac{\tau_x + \tau_y}{2} + \tau_z\right)$  do not bring the system of axes of  $Q^6$  to self-coincidence.

§ 5. Now suppose the translation-group to be  $\Gamma_t'$ .

From  $Q^7$  we have

$$D_{2d}^9 = \{Q^7, S_d\}.$$

Subgroup  $C_{2v}^{20}$ ; 4-al axes  $A, A_1$ .

$$D_{2d}^{10} = \{Q^7, S_d(\tau_z)\}.$$

Subgroup  $C_{2v}^{21}$ ; 4-al axes  $A_1''', A'''$ .

Operations in the plane  $\sigma_{d_1}$  and

$$S_d\left(\frac{\tau_x + \tau_y}{2}\right), S_d\left(\frac{\tau_x + \tau_y}{2} + \tau_z\right)$$

do not bring the system of axes of  $Q^7$  to self-coincidence.



parallel to  $c$  and passing through  $G_2$ ) is a screw-axis of the group of angle  $\pi$  and translation  $2t$ .

There can be no axis not belonging to one of the series  $e, f, g_1, g_2$ , obtained by transforming  $e, f, g_1, g_2$  respectively by the translations of the group. Putting  $t = 0, \frac{\tau_z}{2}, \tau_z, \frac{3\tau_z}{2}$  in turn, we have

$$C_4^1 = \{A_{\frac{\pi}{2}}, \Gamma_t\}. \quad \text{Subgroup } C_2^1.$$

$$C_4^2 = \{A_{\frac{\pi}{2}}, \tau_z, \Gamma_t\}. \quad \text{Subgroup } C_2^2.$$

$$C_4^3 = \{A_{\frac{\pi}{2}}, \tau_z, \Gamma_t\}. \quad \text{Subgroup } C_2^1.$$

$$C_4^4 = \{A_{\frac{\pi}{2}}, \frac{3\tau_z}{2}, \Gamma_t\}. \quad \text{Subgroup } C_2^2.$$

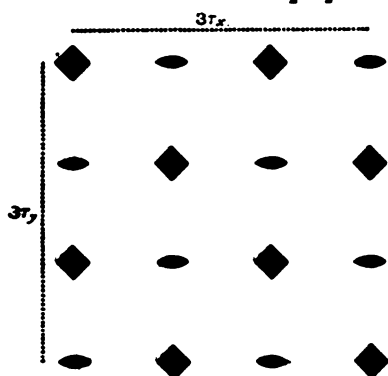


Fig. 180.  $C_4^1$ .

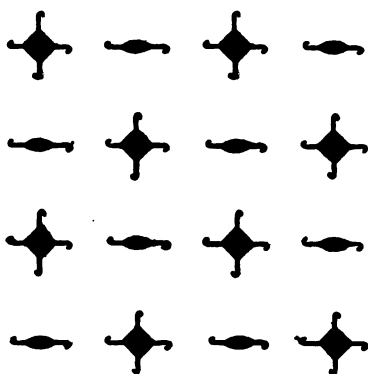


Fig. 181.  $C_4^2$ .

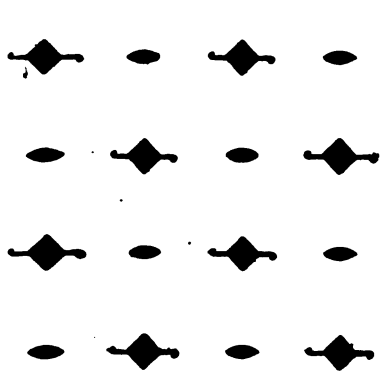


Fig. 182.  $C_4^3$ .

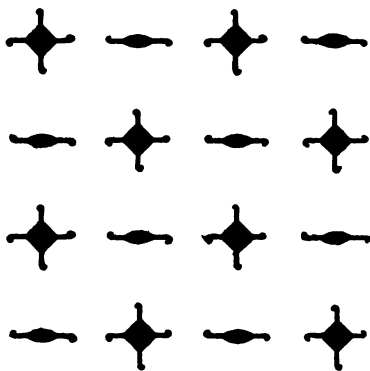


Fig. 183.  $C_4^4$ .

Note that  $e$  and  $f$  may be regarded as 2-al rotation-axes if  $t = 0$  or  $\tau_z$ , and as screw-axes of angle  $\pi$  and translation  $\tau_z$  if  $t = \frac{\tau_z}{2}$  or  $\frac{3\tau_z}{2}$ .

The arrangement of the axes of these groups is shown in Figs. 130 to 133;  $\blacklozenge$  denotes a 4-al rotation-axis, and  $\blacklozenge$ ,  $\blacklozenge$ , screw-axes of angle  $\frac{\pi}{2}$  and translations  $\frac{\tau_z}{2}$ ,  $\tau_z$ ,  $\frac{3\tau_z}{2}$  respectively. The only difference between  $C_2^2$  and  $C_4^4$  is that the screws of one are left-handed and of the other are right-handed.

§ 7. Now let the translation-group be  $\Gamma_t'$

(primitive triplet,  $\tau_y + \tau_z, \tau_z + \tau_x, \tau_x + \tau_y$ ).

Take now  $EE_1 = \tau_x, EE_2 = \tau_y$  (not  $2\tau_x, 2\tau_y$ ) in Fig. 129. Then if the axis  $e$  of an operation  $A_{\frac{\pi}{2}, t}$  passes through  $E$  an axis  $e_3$  similar to  $e$  passes through  $E_3$ , for  $e_3$  is the result of transforming  $e$  by  $\tau_x + \tau_y$ . The group has an operation  $B_{\frac{\pi}{2}, t + \tau_z}$  about  $f$ , for this is equivalent to  $A_{\frac{\pi}{2}, t}$  followed by  $\tau_x + \tau_z$ . Similarly, it has operations  $C_{\pi, 2t + \tau_x}$  about  $g_1$  and  $g_2$ .

Hence the axes of the group are a series (e) of screw-axes of angle  $\frac{\pi}{2}$  and translation  $t$ , a series (f) of screw-axes of angle  $\frac{\pi}{2}$  and translation  $t + \tau_z$ , and two series ( $g_1$  and  $g_2$ ) of axes of angle  $\pi$  and translation  $2t + \tau_z$  derived from  $e, f, g_1$ , and  $g_2$  by transforming by the translations.

A group  $C_4^m$  whose translation-group is  $\Gamma_t'$ , and which contains screws of angle  $\frac{\pi}{2}$  and translation  $t$ , contains also screws of angle  $\frac{\pi}{2}$  and translation  $t + \tau_z$ ; hence  $\{A_{\frac{\pi}{2}, t}, \Gamma_t'\}$  and  $\{A_{\frac{\pi}{2}, t + \tau_z}, \Gamma_t'\}$  are identical.

We have therefore only the two groups

$$C_4^5 = \{A_{\frac{\pi}{2}, 0}, \Gamma_t'\}. \quad \text{Subgroup } C_2^3.$$

$$C_4^6 = \{A_{\frac{\pi}{2}, \tau_z}, \Gamma_t'\}. \quad \text{Subgroup } C_2^3.$$

The arrangement of the axes is shown in Figs. 134, 135.

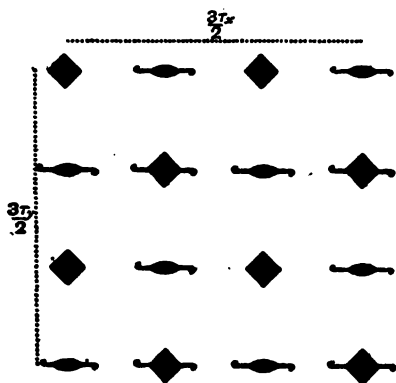


Fig. 134.  $C_4^5$ .

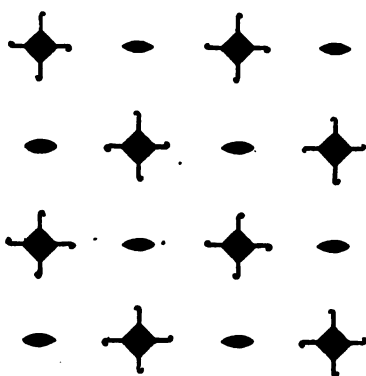


Fig. 135.  $C_4^6$ .

### § 8. TETRAGONAL PARAMORPHY.

Since  $C_{4h}$  is derived from  $C_4$  by multiplying by an inversion, all groups  $C_{4h}^m$  are derived from those groups  $C_4^m$  which satisfy the condition of p. 165—that is, from  $C_4^1, C_4^3, C_4^5, C_4^6$ —by multiplying by an inversion which brings the system of axes to self-coincidence.

To fulfil this last condition the inversion must take place about a point which lies in an axis or halfway between two neighbouring 4-al axes\*. We denote the inversion in these two cases by  $I$  and  $I_1$  respectively.

If the group  $C_{4h}^m$  has a centre of symmetry in one axis it has centres of symmetry in every axis.

We have then

$$C_{4h}^1 = \{C_4^1, I\}. \quad \text{Subgroup } C_{2h}^1.$$

$$C_{4h}^3 = \{C_4^3, I\}. \quad \text{Subgroup } C_{2h}^1.$$

$$C_{4h}^5 = \{C_4^1, I_1\}. \quad \text{Subgroup } C_{2h}^4.$$

$$C_{4h}^6 = \{C_4^3, I_1\}. \quad \text{Subgroup } C_{2h}^4.$$

$$C_{4h}^5 = \{C_4^5, I\}. \quad \text{Subgroup } C_{2h}^3.$$

$$C_{4h}^6 = \{C_4^6, I_1\}. \quad \text{Subgroup } C_{2h}^3.$$

\* That is, at the middle point of  $EF$ , Fig. 129.

There are no more groups, for  $I_1$  does not bring the system of axes of  $C_4^5$  to self-coincidence, nor  $I$  that of  $C_4^6$ . The arrangement of the symmetry- and gliding-planes is given by the consideration of the subgroups  $C_{2k}^m$ .

### § 9. TETRAGONAL HEMIMORPHY.

Since  $C_{4v}$  is derived from  $C_4$  by multiplying by a reflexion in a plane parallel to the 4-al axis, therefore groups  $C_{4v}^m$  are derived from those groups  $C_4^m$  which satisfy the condition of p. 165—that is, from  $C_4^1, C_4^3, C_4^5, C_4^6$ —by multiplying by an operation  $S(t)$  whose plane  $\sigma$  is parallel to the axes and which brings the system of axes to self-coincidence.

Since  $C_{4v}$  has four subgroups  $C_v$ , whose planes make angles of  $\frac{\pi}{4}$  with one another,  $C_{4v}^m$  has four subgroups  $C_v^m$ , whose planes intersect at the same angle. Not all these subgroups can be such that their planes are parallel to two translations of some primitive triplet and perpendicular to the third; therefore at least one of these subgroups is either  $C_2^3$  or  $C_2^4$ , and each group  $C_{4v}^m$  has a symmetry-plane or a gliding-plane whose translation is  $\tau_z$ .

We may take then  $t = 0$  or  $\tau_z$ ; and in that case, since  $S(t)$  brings the system of axes of  $C_4^m$  to self-coincidence,  $\sigma$  joins two neighbouring axes of the series  $e$  and  $f$ , two neighbouring axes of the series  $e$  and  $g$ ,\*, two neighbouring axes of the series  $f$  and  $g_2$ , or two neighbouring axes of the series  $g_1$  and  $g_2$ . We shall denote  $\sigma$  in these cases by  $\sigma_s, \sigma_a, \sigma_b$ , and  $\sigma_c$  respectively (Fig. 129), and the corresponding operations by  $S_s(t), S_a(t), S_b(t)$ , and  $S_c(t)$  respectively ( $t = 0$  or  $\tau_z$ ).  $\sigma_a$  and  $\sigma_b$  are only geometrically distinguishable in the case of  $C_4^5$  and  $C_4^6$ .

First suppose the translation-group is  $\Gamma_7$ .

We have then

$$C_{4v}^1 = \{C_4^1, S_s\} = \{C_4^1, S_a\} \dagger.$$

$$C_{4v}^2 = \{C_4^1, S_c\}.$$

$$C_{4v}^3 = \{C_4^3, S_s\} = \{C_4^3, S_a(\tau_z)\}.$$

$$C_{4v}^4 = \{C_4^3, S_c\}.$$

\* The case in which  $\sigma$  joins two neighbouring axes of the series  $e$  and  $g$ , is not geometrically distinguishable from this.

† The group  $\{C_4^1, S_s\}$  has  $\sigma_a$  as a symmetry-plane; therefore by p. 165  $\{C_4^1, S_s\}$  and  $\{C_4^1, S_a\}$  are identical.

$$C_{4v}^3 = \{C_4^1, S_8(\tau_z)\} = \{C_4^1, S_a(\tau_z)\}.$$

$$C_{4v}^3 = \{C_4^1, S_o(\tau_z)\}.$$

$$C_{4v}^1 = \{C_4^3, S_8(\tau_z)\} = \{C_4^3, S_a\}.$$

$$C_{4v}^3 = \{C_4^3, S_o(\tau_z)\}.$$

The arrangement of the axes and planes of these groups is shown in Figs. 136 to 143.

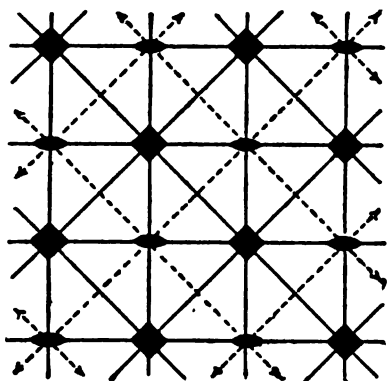


Fig. 136.  $C_{4v}^1$ .

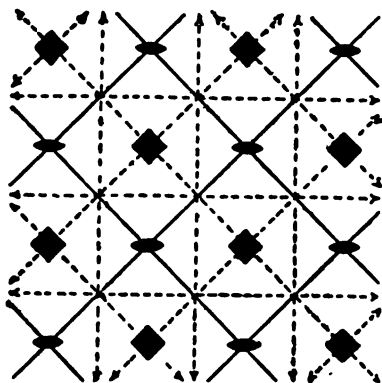


Fig. 137.  $C_{4v}^2$ .

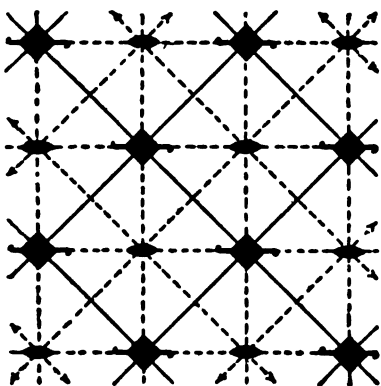


Fig. 138.  $C_{4v}^3$ .

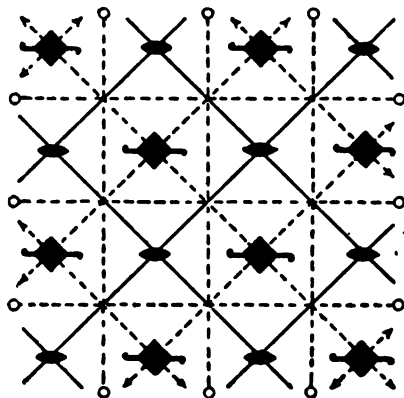
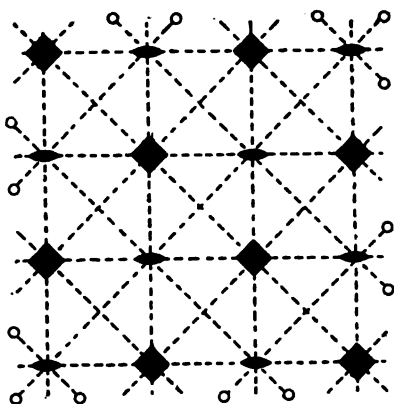
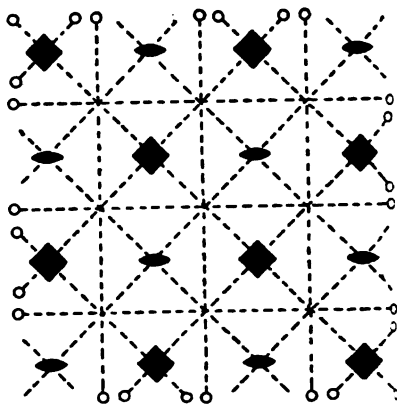
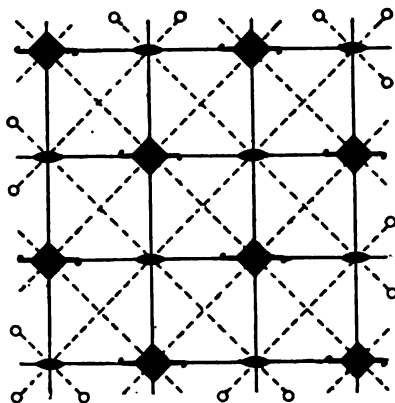
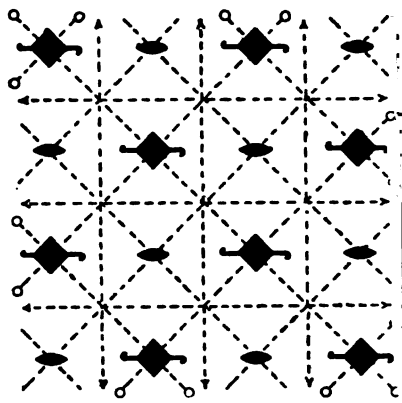


Fig. 139.  $C_{4v}^4$ .

Fig. 140.  $C_{4v}^5$ .Fig. 141.  $C_{4v}^8$ .Fig. 142.  $C_{4v}^7$ .Fig. 143.  $C_{4v}^6$ .

§ 10. If the translation-group is  $\Gamma_t'$  we have the following groups (Figs. 144 to 147)\*.

$$C_{4v}^9 = \{C_4^5, S_a\} = \{C_4^5, S_a\} = \{C_4^5, S_b(\tau_s)\}.$$

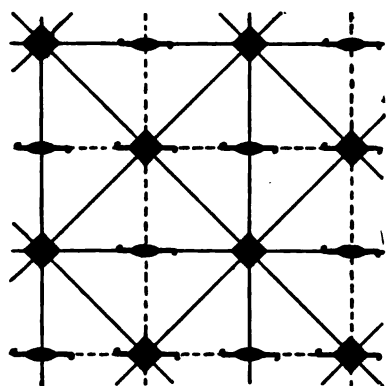
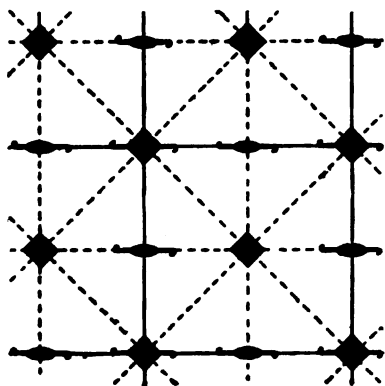
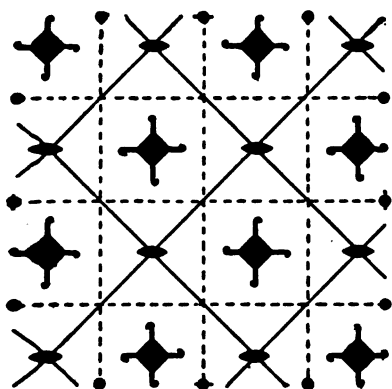
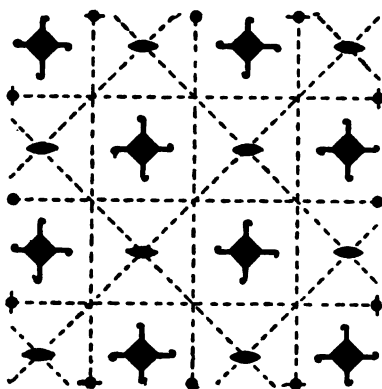
$$C_{4v}^{10} = \{C_4^5, S_a(\tau_s)\} = \{C_4^5, S_b\} = \{C_4^5, S_a(\tau_s)\}.$$

$$C_{4v}^{11} = \{C_4^6, S_c\}.$$

$$C_{4v}^{12} = \{C_4^6, S_c(\tau_s)\}.$$

\* In these figures a broken line means a plane of translation  $\tau_t$ , a broken line with dark circles means a plane of translation  $\frac{1}{2}(\tau_s + \tau_s)$  or  $\frac{1}{2}(\tau_y + \tau_s)$ , and one with dark circles with lines through them means a plane of translation  $\frac{1}{2}(\tau_s - \tau_s)$  or  $\frac{1}{2}(\tau_y - \tau_s)$ .

There are no more groups  $C_{4v}^m$ , for  $S_c, S_c(\tau_z)$  do not bring the system of axes of  $C_4^5$  to self-coincidence, nor  $S_a, S_a(\tau_z), S_b, S_b(\tau_z)$  the system of  $C_4^{*6}$ .

Fig. 144.  $C_{4v}^9$ .Fig. 145.  $C_{4v}^{10}$ .Fig. 146.  $C_{4v}^{11}$ .Fig. 147.  $C_{4v}^{12}$ .

### § 11. TETRAGONAL ENANTIOMORPHY.

Since the point-group  $D_4$  is derived from  $C_4$  by multiplying by a rotation about a 2-al axis perpendicular to the axis of  $C_4$ , therefore the groups  $D_4^m$  are derived from  $C_4^m$  by multiplying

\* It will be remembered that an operation of the second sort must bring right-handed screw-axes into coincidence with left-handed and vice versa (p. 156).

by a rotation or screw of angle  $\pi$  about an axis perpendicular to the axes of  $C_4^m$  which brings the system formed by these axes to self-coincidence. Because  $D_4$  has four subgroups  $C_2$ , therefore  $D_4^m$  has four subgroups  $C_2^m$ ; by reasoning similar to that of § 9 it may be shown that one of these subgroups at least is  $C_2^s$ . Hence every group  $D_4^m$  has a rotation of angle  $\pi$ . The axis of such a rotation must evidently have a position similar either to  $EF$ ,  $EG_1$ ,  $FG_2$ , or  $G_1G_2$  of Fig. 129 (p. 202). We shall denote these lines by  $u_s$ ,  $u_a$ ,  $u_b$ ,  $u_c$  respectively, and rotations through  $\pi$  about them by  $U_s$ ,  $U_a$ ,  $U_b$ ,  $U_c$ .

However, the existence of the translation  $2\tau_y$  in the case of  $\Gamma_t$  and of  $\tau_y + \tau_z$  in the case of  $\Gamma_t'$  shows that  $\{C_4^m, U_a\}$  and  $\{C_4^m, U_b\}$  are always identical.

§ 12. Let the translation-group be  $\Gamma_t$ .

We have then

$$\begin{aligned} D_4^1 &= \{C_4^1, U_s\} = \{C_4^1, U_a\}, \\ D_4^2 &= \{C_4^1, U_c\}. \end{aligned}$$

The arrangement of the axes of these groups in each member of a set of principal planes of interval  $\tau_z$  is shown in Figs. 148, 149; 2-al rotation-axes are represented by thick lines, screw-axes by thick broken lines.

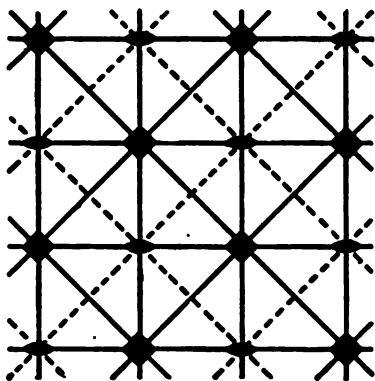


Fig. 148.  $D_4^1$ .

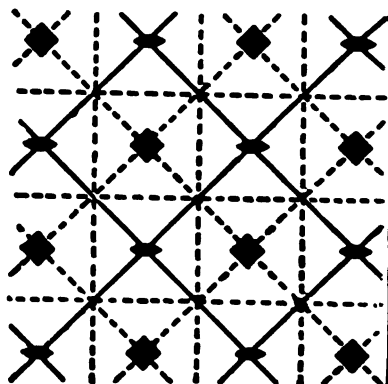


Fig. 149.  $D_4^2$ .

A similar arrangement holds for

$$\begin{aligned} D_4^3 &= \{C_4^2, U_s\} = \{C_4^2, U_a\}, \\ D_4^4 &= \{C_4^2, U_c\}. \end{aligned}$$

except that the axes of the subgroups  $C_2^m$  do not intersect. Axes parallel to  $EG_1$  lie in each member of a set of principal planes of interval  $\tau_z$ ; axes parallel to  $EF$ ,  $EG_2$ ,  $G_1G_2$  lie in the planes obtained by transforming this set by  $\frac{\tau_z}{4}$ ,  $\frac{\tau_z}{2}$ ,  $\frac{3\tau_z}{4}$  respectively.

The same holds for

$$\begin{aligned} D_4^7 &= \{C_4^4, U_s\} = \{C_4^4, U_a\}, \\ D_4^8 &= \{C_4^4, U_o\}, \end{aligned}$$

if we substitute  $-\tau_z$  for  $\tau_z$ .

For 
$$\begin{aligned} D_4^5 &= \{C_4^3, U_s\} = \{C_4^3, U_a\}, \\ D_4^6 &= \{C_4^3, U_o\} \end{aligned}$$

axes parallel to  $EG_1$  and  $EG_2$  lie in each member of a set of principal planes of interval  $\tau_z$ ; axes parallel to  $EF$  and  $G_1G_2$  lie in the planes halfway between any two consecutive members of this set.

§ 13.  $U_c$  does not bring the system of axes of  $C_4^5$  and  $C_4^6$  whose translation-group is  $\Gamma'_t$  into self-coincidence.

We have then only

$$\begin{aligned} D_4^9 &= \{C_4^5, U_s\} = \{C_4^5, U_a\}, \\ D_4^{10} &= \{C_4^6, U_s\} = \{C_4^6, U_a\}, \end{aligned}$$

$D_4^9$  has 2-al axes as in Fig. 150 in each of a set of principal planes of interval  $\tau_z$ ; it has also axes as in Fig. 151 in the planes halfway between any two consecutive members of this set.

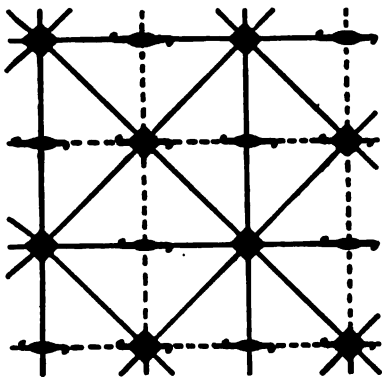


Fig. 150.  $D_4^9$  (1).

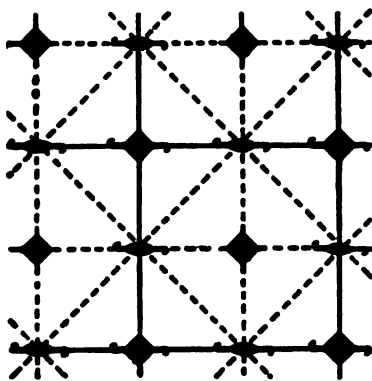
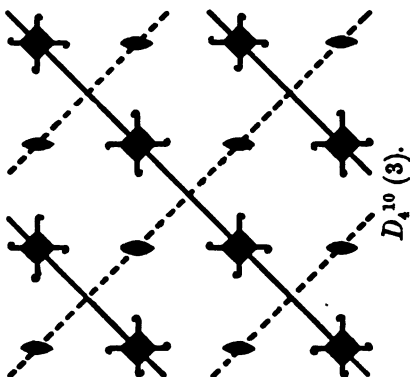
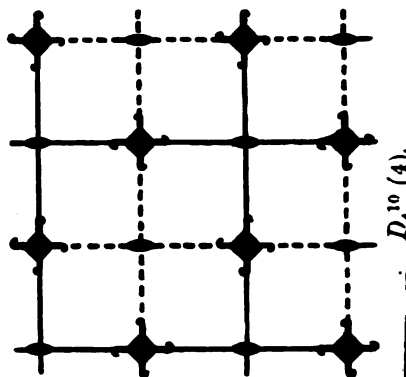


Fig. 151.  $D_4^9$  (2).

$D_4^{10}$  has axes as shown in Fig. 152 in each member of a set of principal planes of interval  $\tau_z$ ; and as shown in Fig. 153 in the planes obtained by transforming the set by  $\frac{\tau_z}{4}$ . The arrangement of the axes in the planes obtained by transforming the set by  $\frac{\tau_z}{2}$  and  $\frac{3\tau_z}{4}$  is given by turning Figs. 152 and 153 respectively through a right angle.

Fig. 152.  $D_4^{10}$  (1).Fig. 153.  $D_4^{10}$  (2).

#### § 14. TETRAGONAL HOLOHEDRY.

$D_{4h}$  is derived from  $D_4$  by multiplying by an inversion; therefore the groups  $D_{4h}^m$  are derived from those groups  $D_4^m$  which satisfy the condition of p. 165—that is,  $D_4^1$ ,  $D_4^2$ ,  $D_4^5$ ,  $D_4^6$ ,  $D_4^9$ ,  $D_4^{10}$ —by multiplying by an inversion (about a point  $P$ ) which brings the system of axes to self-coincidence.

This last condition is evidently only fulfilled if the inversion brings the system of axes of the subgroup  $C_4^m$  and of each of the subgroups  $C_2^m$  to self-coincidence; conversely it is fulfilled if the inversion brings the system of axes of  $C_4^m$  and of one of the subgroups  $C_2^m$ —say that one whose axes are parallel to  $EF(u_e)$  of Fig. 129—into self-coincidence (p. 166). It is sufficient to investigate the cases in which  $P$  is on the axis  $e$  or halfway between  $e$  and  $f$ ; for each group  $C_{4h}^m$  has a centre of symmetry in one or other of these positions, and each group  $D_{4h}^m$  has a subgroup  $C_{4h}^m$ .

§ 15. First, let the translation-group be  $\Gamma_t$ . In groups derived from  $D_4^1$ ,  $D_4^2$ ,  $D_4^5$ , and  $D_4^6$   $P$  may evidently be at the point of intersection of  $e$  and an axis parallel to  $u_e$ , halfway between two such points of intersection, on an axis parallel to

$u_s$  halfway between  $e$  and  $f$ , or halfway between  $e$  and  $f$  and halfway between axes parallel to  $u_s$ ; for in such positions and in such only is it a centre of symmetry for the system of axes of  $C_4^m$  and the system of that subgroup  $C_2^m$  whose axes are parallel to  $u_s$ . The inversions in the four cases are denoted by  $I, I', I_1$ , and  $I_1'$  respectively.

We have therefore:—

$$D_{4h}^1 = \{D_4^1, I\}. \quad \text{Subgroups } C_{4h}^1; C_{4v}^1.$$

$$D_{4h}^2 = \{D_4^1, I'\}. \quad \text{Subgroups } C_{4h}^1; C_{4v}^5.$$

$$D_{4h}^3 = \{D_4^1, I_1\}. \quad \text{Subgroups } C_{4h}^3; C_{4v}^3.$$

$$D_{4h}^4 = \{D_4^1, I_1'\}. \quad \text{Subgroups } C_{4h}^3; C_{4v}^6.$$

$$D_{4h}^5 = \{D_4^2, I\}. \quad \text{Subgroups } C_{4h}^1; C_{4v}^2.$$

$$D_{4h}^6 = \{D_4^2, I'\}. \quad \text{Subgroups } C_{4h}^1; C_{4v}^6.$$

$$D_{4h}^7 = \{D_4^2, I_1\}. \quad \text{Subgroups } C_{4h}^3; C_{4v}^1.$$

$$D_{4h}^8 = \{D_4^2, I_1'\}. \quad \text{Subgroups } C_{4h}^3; C_{4v}^5.$$

$$D_{4h}^{10} = \{D_4^5, I\}^*. \quad \text{Subgroups } C_{4h}^2; C_{4v}^3.$$

$$D_{4h}^9 = \{D_4^5, I'\}. \quad \text{Subgroups } C_{4h}^2; C_{4v}^6.$$

$$D_{4h}^{12} = \{D_4^5, I_1\}. \quad \text{Subgroups } C_{4h}^4; C_{4v}^4.$$

$$D_{4h}^{11} = \{D_4^5, I_1'\}. \quad \text{Subgroups } C_{4h}^4; C_{4v}^5.$$

$$D_{4h}^{14} = \{D_4^6, I\}. \quad \text{Subgroups } C_{4h}^2; C_{4v}^4.$$

$$D_{4h}^{13} = \{D_4^6, I'\}. \quad \text{Subgroups } C_{4h}^2; C_{4v}^5.$$

$$D_{4h}^{16} = \{D_4^6, I_1\}. \quad \text{Subgroups } C_{4h}^4; C_{4v}^3.$$

$$D_{4h}^{15} = \{D_4^6, I_1'\}. \quad \text{Subgroups } C_{4h}^4; C_{4v}^6.$$

§ 16. Now let the translation-group be  $\Gamma'_t$ . In groups derived from  $D_4^9$ ,  $P$  must be either at the point of intersection of  $e$  with an axis parallel to  $u_s$ , or halfway between two such points of intersection.  $P$  cannot lie halfway between  $e$  and  $f$ , for there is no group  $C_{4h}^m$  derived from  $C_4^5$  with a centre of

\* The natural notation has been altered to bring it into accordance with that of Schoenflies. Reference to his "Krystallsysteme und Krystallstruktur" will be thereby facilitated.

symmetry in that position. Similarly in groups derived from  $D_4^{10}$ ,  $P$  must be either halfway between  $e$  and  $f$  and on an axis parallel to  $u_e$ , or halfway between  $e$  and  $f$  and halfway between axes parallel to  $u_e$ .

$$D_{4h}^{17} = \{D_4^9, I\}. \quad \text{Subgroups } C_{4h}^5; C_{4v}^9.$$

$$D_{4h}^{18} = \{D_4^9, I'\}. \quad \text{Subgroups } C_{4h}^5; C_{4v}^{10}.$$

$$D_{4h}^{19} = \{D_4^{10}, I_1\}. \quad \text{Subgroups } C_{4h}^6; C_{4v}^{11}.$$

$$D_{4h}^{20} = \{D_4^{10}, I_1'\}. \quad \text{Subgroups } C_{4h}^6; C_{4v}^{12}.$$

## CHAPTER XXI

## RHOMBOHEDRAL GROUPS.

§ 1. The point-groups of the rhombohedral system all possess one 3-al rotation-axis; therefore the isomorphous space-groups have screws of angle  $\frac{2\pi}{3}$  about a single series of parallel axes. The planes perpendicular to these axes will be called principal planes. We shall take  $2\tau_s$  as the primitive translation parallel to any reduced screw  $A_{\frac{2\pi}{3}, t}$ ; so that  $t = 0, \frac{2\tau_s}{3}, \text{ or } \frac{4\tau_s}{3}$ .

The translation-group of any rhombohedral group is either  $\Gamma_h$  or  $\Gamma_{rh}$  (p. 159). Since  $\Gamma_h$  contains no translation whose component in the direction of  $\tau_s$  is less than  $2\tau_s$ , therefore the translations of all screws of a group whose translation-group is  $\Gamma_h$  are the same. On the other hand  $\Gamma_{rh}$  contains a translation whose component parallel to  $\tau_s$  is  $\frac{2\tau_s}{3}$ ; therefore if a group whose translation-group is  $\Gamma_{rh}$  has screws whose translation is  $t$ , it has screws whose translations are  $t + \frac{2\tau_s}{3}$  and  $t + \frac{4\tau_s}{3}$ ; that is, it has screws whose translations are  $0, \frac{2\tau_s}{3}, \text{ and } \frac{4\tau_s}{3}$ , since  $t$  has one of the values  $0, \frac{2\tau_s}{3}, \frac{4\tau_s}{3}$ .

The series of axes obtained by transforming any axis  $e$  by the translations will (as before) be denoted by  $\mathfrak{e}$ .

## § 2. RHOMBOHEDRAL TETARTOHEDRY.

Groups of this class have a single series of parallel 3-al axes; their operations are obtained by multiplying the operations of the translation-group by a single operation of the form  $A_{\frac{2\pi}{3}, t}$ .

First, let  $\Gamma_h$  be the translation-group; all screws of such groups  $C_3^m$  have the same translation (p. 161).

Suppose the axis  $e$  of  $A_{\frac{2\pi}{3}, t}$  to be perpendicular to the plane of the paper and to meet it in  $E$ . Take  $EE_1 = 2\tau_1$ ,  $EE_2 = 2\tau_2$ ,  $EE_3 = -2\tau_1 - 2\tau_2$ ,  $EE_4 = 2\tau_1 + 2\tau_2$ ,  $EE_5 = -2\tau_1$ ,  $EE_6 = -2\tau_2$ , using the notation of p. 139 (Fig. 154).

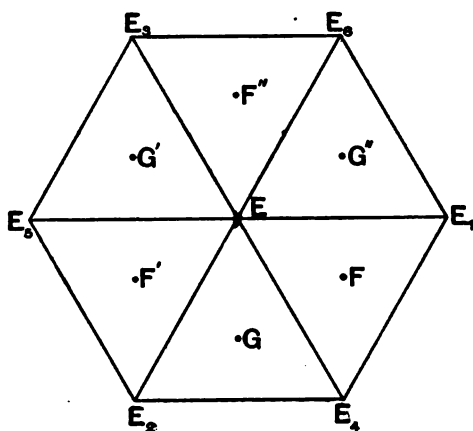


Fig. 154.

Then, since  $EE_1$ ,  $EE_2$ ,  $EE_3$ ,  $EE_4$ ,  $EE_5$ ,  $EE_6$  are translations of the group

$$\{A_{\frac{2\pi}{3}, t}, \Gamma_h\},$$

therefore axes similar to  $e$  pass through the points  $E_1$ ,  $E_2$ ,  $E_3$ ,  $E_4$ ,  $E_5$ ,  $E_6$ .

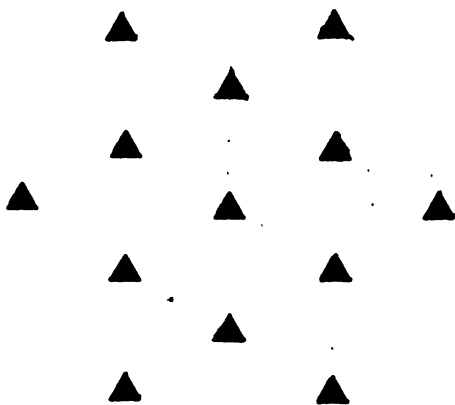
Since the operation  $A_{\frac{2\pi}{3}, t}$  followed

by the translation  $EE_1$  is equivalent to a screw  $B_{\frac{2\pi}{3}, t}$  about

an axis  $f$ , parallel to  $e$ , passing through the centre  $F$  of the triangle  $EE_1E_4$  (p. 148), therefore the group has  $f$  as a screw-axis; similarly it has a screw  $C_{\frac{2\pi}{3}, t}$  about an axis  $g$

passing through the centre of the triangle  $EE_2E_5$ . It has no axes other than those obtained by transforming  $e, f, g$  by the translations of  $\Gamma_h$ ; for if it had a screw

$H_{\frac{2\pi}{3}, t}$  about any other axis the group would have a translation  $A_{\frac{2\pi}{3}, t} \cdot (H_{\frac{2\pi}{3}, t})^{-1}$ —not belonging to  $\Gamma_h$  which is impossible.

Fig. 155.  $C_3^1$ .

We have then the three groups:—

$$C_3^1 = \{A_{\frac{2\pi}{3}}, \Gamma_h\}.$$

$$C_3^2 = \{A_{\frac{2\pi}{3}}, \frac{2\tau_z}{3}, \Gamma_h\}.$$

$$C_3^3 = \{A_{\frac{2\pi}{3}}, \frac{4\tau_z}{3}, \Gamma_h\}.$$

The group  $C_3^3$  contains the operation  $A_{\frac{2\pi}{3}}, -\frac{2\tau_z}{3}$ ; for this is equivalent to

$$A_{\frac{2\pi}{3}}, \frac{4\tau_z}{3},$$

followed by the translation  $-2\tau_z$ . Therefore the only difference between  $C_3^2$  and  $C_3^3$  is that the screws of one are left-handed and those of the other right-handed.

The arrangement of the axes of the groups  $C_3^1, C_3^2, C_3^3$  is shown in Figs. 155, 156, 157; a 3-al rotation-axis is represented by  $\blacktriangle$ , and 3-al screw-axes of translation

$$\frac{2\tau_z}{3}, \frac{4\tau_z}{3}$$

by  $\blacktriangleleft, \blacktriangleright$ , respectively.

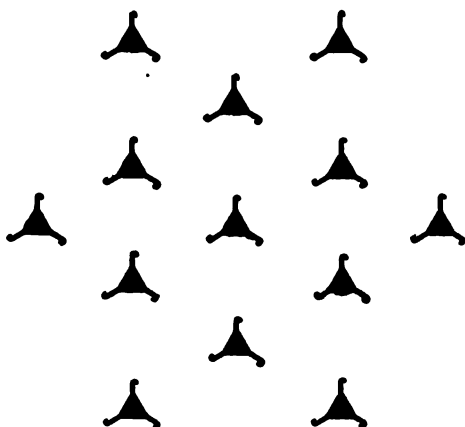


Fig. 156.  $C_3^2$ .

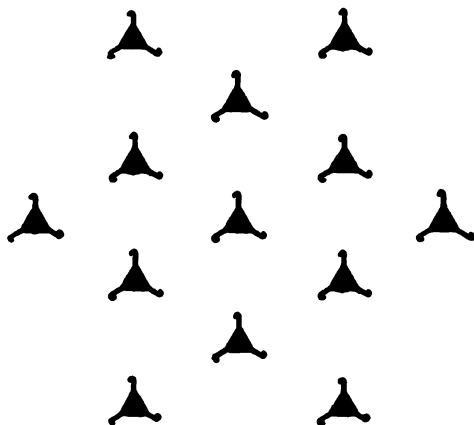


Fig. 157.  $C_3^3$ .

§ 3. Now suppose the translation-group to be  $\Gamma_h$ . Take any point  $O$  in the axis  $e$  of a screw  $A_{\frac{2\pi}{3}, \frac{2\tau_z}{3}}$ , and let  $OE_1, OE_2, OE_3$  represent the translations  $2\tau', 2\tau'', 2\tau'''$  of p. 141.

Let the plane  $E_1E_2E_3$  cut  $e$  in  $E$  (Figs. 158, 159).

Then axes exactly similar to  $e$  pass through  $E_1, E_2, E_3$ . Now the screw  $A_{\frac{2\pi}{3}, t}$ , followed by the translation  $OE_1$  (whose component parallel to  $e$  is  $\frac{2\tau_z}{3}$ ), is equivalent to a screw

$B_{\frac{2\pi}{3}, t + \frac{2\tau_z}{3}}$  about an axis  $f$ , passing through the point  $F$  a third of the way from  $E_1$  to  $E_2$ ; the group

$$\{A_{\frac{2\pi}{3}, t}, \Gamma_{rh}\}$$

has therefore  $f$ , as a screw-axis, whose translation is  $t + \frac{2\tau_z}{3}$ ; similarly it has a screw-axis  $g$ , whose translation is  $t + \frac{4\tau_z}{3}$ ,

passing through the point  $G$  halfway between  $F$  and  $E_2$ . As before it may be seen that the group has no axes except those obtained by transforming  $e, f, g$  by the translations. It is evident that the axes  $e, f, g$  have exactly similar positions with regard to all the elements of the group, so that it is indifferent whether we take

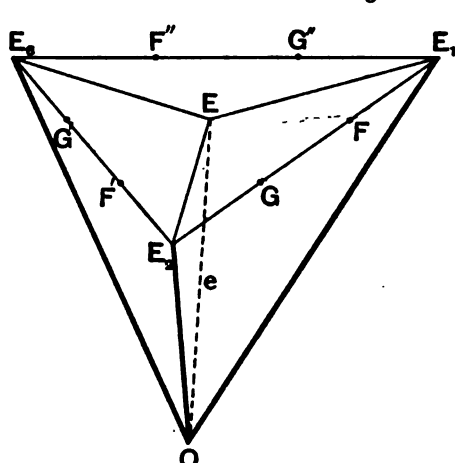


Fig. 158.

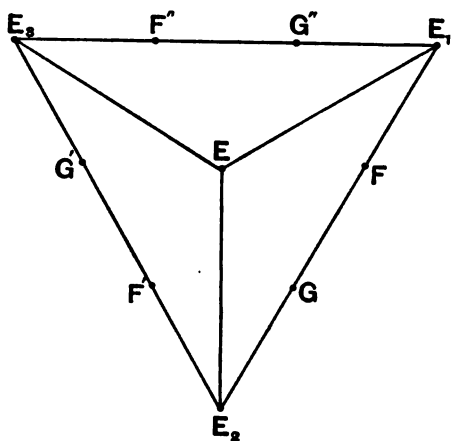


Fig. 159.

$$t = 0 \text{ (and therefore } t + \frac{2\tau_z}{3} = \frac{2\tau_z}{3}, t + \frac{4\tau_z}{3} = \frac{4\tau_z}{3})$$

$$\text{or } t = \frac{2\tau_z}{3} \text{ (and therefore } t + \frac{2\tau_z}{3} = \frac{4\tau_z}{3}, t + \frac{4\tau_z}{3} - 2\tau_z = 0)$$

or  $t = \frac{4\tau_z}{3}$  (and therefore  $t + \frac{2\tau_z}{3} - 2\tau_z = 0$ ,  $t + \frac{4\tau_z}{3} - 2\tau_z = \frac{2\tau_z}{3}$ ).

Hence we have only one group of this sort (Fig. 160).

$$C_3^4 = \{A_{\frac{2\pi}{3}, 0}, \Gamma_{rh}\} = \{A_{\frac{2\pi}{3}, \frac{2\tau_z}{3}}, \Gamma_{rh}\} = \{A_{\frac{2\pi}{3}, \frac{4\tau_z}{3}}, \Gamma_{rh}\}.$$

The only difference between the series of axes  $f$  and  $g$  is that one series is left-handed and the other right-handed.

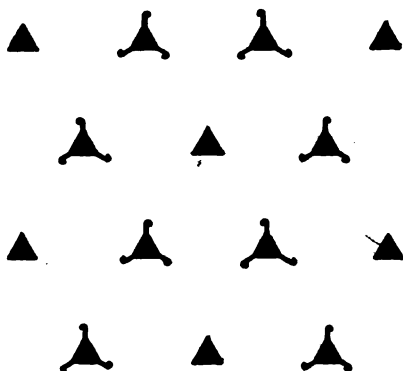


Fig. 160.  $C_3^4$ .

#### § 4. HEXAGONAL TETARTOEDRY OF THE SECOND SORT.

Since the group  $C_{3i}$  is obtained by multiplying the operations of  $C_3$  by an inversion, therefore all groups  $C_{3i}^m$  are obtained by multiplying the operations of those groups  $C_3^m$  which satisfy the condition of p. 165—that is,  $C_3^1$  and  $C_3^4$ —by an inversion which brings the system of axes of  $C_3^m$  into self-coincidence. In order that this last condition may be fulfilled the inversion must take place about a point in an axis or halfway between two axes; if, however, a group  $C_3^m$  has a centre of symmetry lying in an axis, it has one lying halfway between two axes and vice versa. For if  $E$  (Fig. 154) is a centre of symmetry the point midway between  $F$  and  $G$  must also be a centre of symmetry, since  $EE_1$  is a translation; and a similar proof holds in the case of  $C_3^4$ . It is therefore sufficient to combine  $C_3^1$  and  $C_3^4$  with an inversion about a point in an axis.

All axes of  $C_3^1$  are similar, and therefore we have only one group derivable from  $C_3^1$ .

$$C_{3i}^1 = \{C_3^1, I\}.$$

The axes of  $C_4^3$  are not similar, but inversion about a point in the axis  $f$  or  $g$  does not bring the system of axes to self-coincidence. Hence the only group derivable from  $C_3^4$  is obtained by combining its operations with inversion about a point in a *rotation-axis*.

$$C_{3i}^3 = \{C_3^4, I\}.$$

### § 5. RHOMBOHEDRAL HEMIMORPHY.

Exactly as in § 9 of the previous chapter it may be shown that the groups  $C_{3\sigma}$  are derived from  $C_3^1$  or  $C_3^4$  by multiplying by an operation  $S$  or  $S(\tau_z)$  in a plane  $\sigma$  parallel to the axes, which brings the system of axes to self-coincidence.

This last condition is evidently only satisfied if  $\sigma$  passes through a line such as  $EF$  or  $EE_3$  of Fig. 154 or a line such as  $EE_3$  of Fig. 159. Planes parallel to the axes through  $EF$  and  $EE_3$  will be denoted by  $\sigma_s$  and  $\sigma_a$  respectively.

We have then the following groups:—

$C_{3\sigma}^1 = \{C_3^1, S_s\}.$	Subgroups $C_s^3. (3)^*.$
$C_{3\sigma}^2 = \{C_3^1, S_a\}.$	Subgroups $C_s^3. (3).$
$C_{3\sigma}^3 = \{C_3^1, S_s(\tau_z)\}.$	Subgroups $C_s^4. (3).$
$C_{3\sigma}^4 = \{C_3^1, S_a(\tau_z)\}.$	Subgroups $C_s^4. (3).$
$C_{3\sigma}^5 = \{C_3^4, S_s\}.$	Subgroups $C_s^3. (3).$
$C_{3\sigma}^6 = \{C_3^4, S_a(\tau_z)\}.$	Subgroups $C_s^4. (3).$

See Figs. 161 to 166.

### § 6. RHOMBOHEDRAL ENANTIOMORPHY.

By reasoning exactly similar to that of § 11 of the previous chapter, it may be shown that all groups  $D_3^m$  are obtained by multiplying  $C_3^m$  by a *rotation* of angle  $\pi$ , which brings the system of axes of  $C_3^m$  to self-coincidence. To fulfil this last condition it is necessary that the axis of this rotation should be such a line as  $EF$  or  $EE_3$  of Fig. 154 or  $EF$  of Fig. 159. We denote rotations of angle  $\pi$  about  $EF$  and  $EE_3$  by  $U_s$  and  $U_a$  respectively.

Since no two members of a primitive pair in a principal

\* That is,  $C_{3i}^1$  has three subgroups  $C_s^3$ , each of which is  $C_s^3$ .

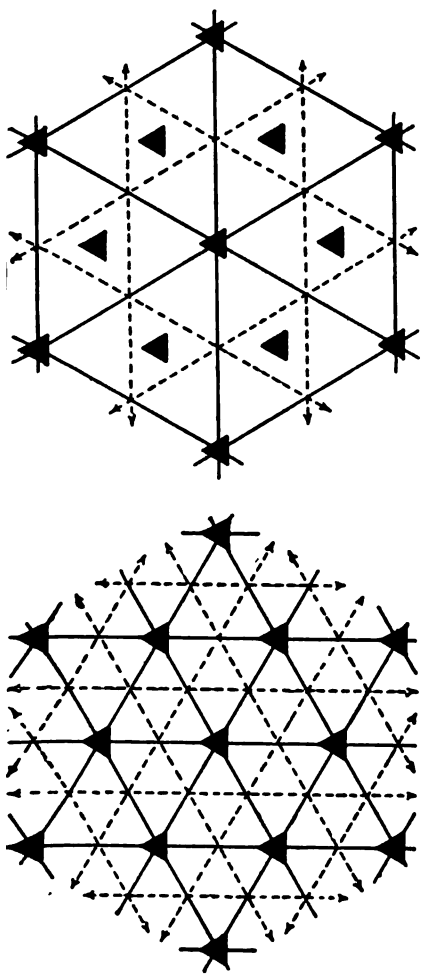


Fig. 161.  $C_{3v}^1$ .

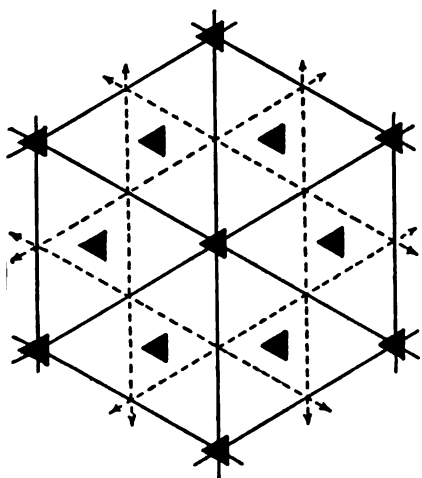


Fig. 162.  $C_{3v}^2$ .

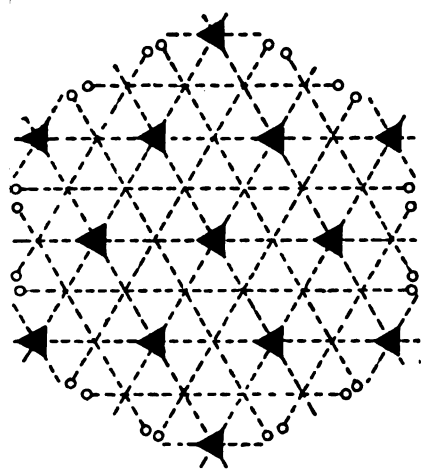


Fig. 163.  $C_{3v}^3$ .

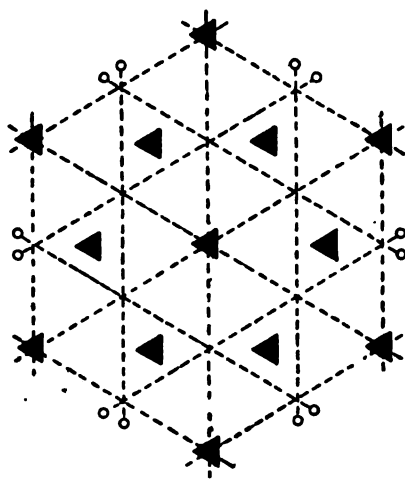


Fig. 164.  $C_{3v}^4$ .

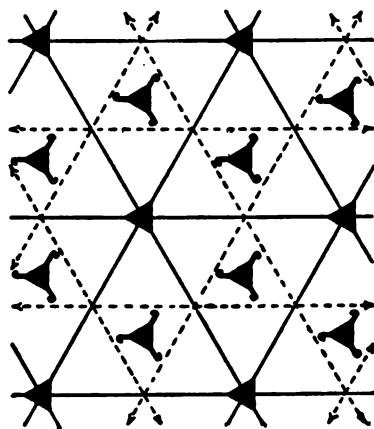


Fig. 165.  $C_{3v}^5$ .

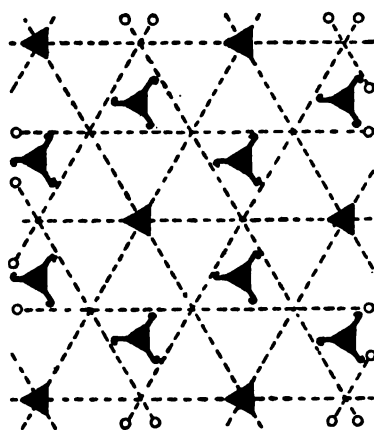


Fig. 166.  $C_{3v}^6$ .

plane are at right angles each of the three subgroups  $C_2^m$  is  $C_2^3$ .

We have then

$$\begin{aligned} D_3^1 &= \{C_3^1, U_s\}, \\ D_3^2 &= \{C_3^1, U_a\}. \end{aligned}$$

These groups have axes arranged as in Figs. 167, 168, respectively, in each member of a set of principal planes of interval  $\tau_z$ .

The arrangement of the axes of

$$D_3^3 = \{C_3^2, U_s\},$$

$$D_3^4 = \{C_3^2, U_a\}$$

is similar, except that the axes of the subgroups  $C_3^3$  do not intersect. Axes parallel to  $EF$  or  $EE_4$  (Fig. 154) lie in each member of a set of principal planes of interval  $\tau_z$ ; axes parallel to  $EF'$  or  $EE_2$  lie in planes obtained by transforming

this set by  $\frac{\tau_z}{3}$ ; axes parallel to  $EF'$  or  $EE_1$  lie in planes obtained by transforming the set by  $\frac{2\tau_z}{3}$ .

To

$$D_3^5 = \{C_3^3, U_s\},$$

$$D_3^6 = \{C_3^3, U_a\}$$

the above remarks apply if we put  $-\tau_z$  for  $\tau_z$ .

The translation-group of the above groups is  $\Gamma_A$ ; the only group whose translation-group is  $\Gamma_{rA}$  is

$$D_3^7 = \{C_3^4, U_s\}.$$

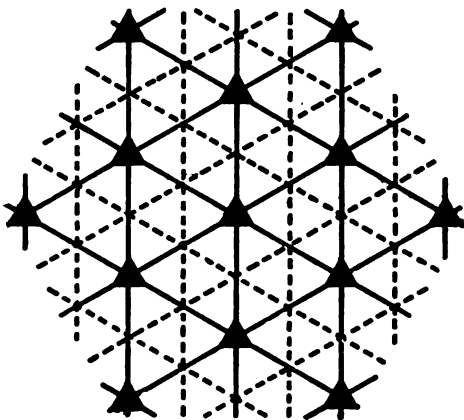


Fig. 167.  $D_3^1$ .

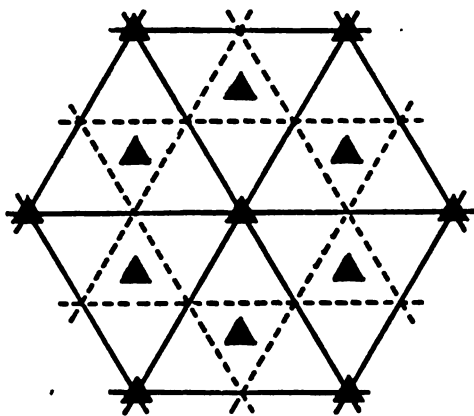
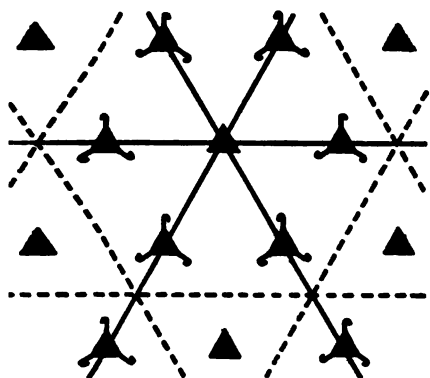
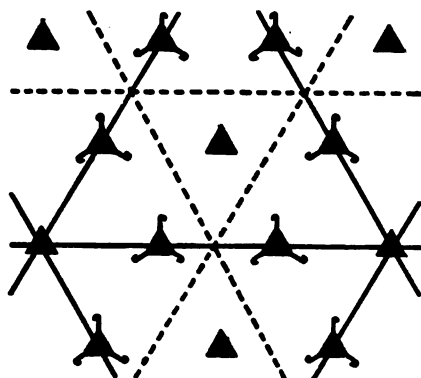
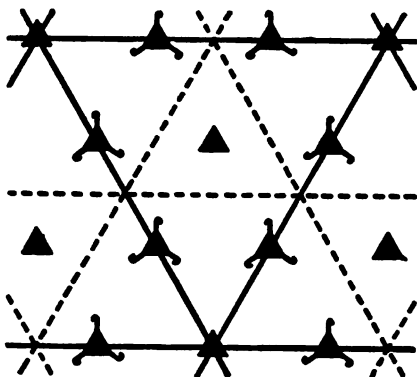


Fig. 168.  $D_3^2$ .

It has axes as shown in Fig. 169 in each member of a set of principal planes of interval  $\tau_z$ ; and as shown in Figs. 170, 171 in the planes obtained by transforming the set by  $\frac{\tau_z}{3}$ ,  $\frac{2\tau_z}{3}$ , respectively.

Fig. 169.  $D_3^1(1)$ .Fig. 170.  $D_3^1(2)$ .Fig. 171.  $D_3^1(3)$ .

## § 7. RHOMBOHEDRAL HOLOHEDRY.

By reasoning very similar to that of § 14 of the previous chapter, it may be shown that all groups  $D_{3d}^m$  are obtained by multiplying those groups  $D_3^m$  which satisfy the condition of p. 165—that is  $D_3^1$ ,  $D_3^2$ , and  $D_3^1$ —by an inversion about a point lying in a 3-al rotation-axis, either at its intersection

with a perpendicular 2-al axis or halfway between two such intersections.

Denoting the inversions in these two cases by  $I$  and  $I'$ , respectively, we have,

$$D_{3d}^1 = \{D_3^1, I\}. \quad \text{Subgroups } C_{3d}^1; C_{3v}^2; C_{2d}^3 (3).$$

$$D_{3d}^2 = \{D_3^1, I'\}. \quad \text{Subgroups } C_{3d}^1; C_{3v}^4; C_{2d}^6 (3).$$

$$D_{3d}^3 = \{D_3^2, I\}. \quad \text{Subgroups } C_{3d}^1; C_{3v}^1; C_{2d}^3 (3).$$

$$D_{3d}^4 = \{D_3^2, I'\}. \quad \text{Subgroups } C_{3d}^1; C_{3v}^3; C_{2d}^6 (3).$$

$$D_{3d}^5 = \{D_3^3, I\}. \quad \text{Subgroups } C_{3d}^2; C_{3v}^5; C_{2d}^3 (3).$$

$$D_{3d}^6 = \{D_3^3, I'\}. \quad \text{Subgroups } C_{3d}^2; C_{3v}^6; C_{2d}^6 (3).$$

The arrangements of the elements of symmetry are given by consideration of the subgroups.

## CHAPTER XXII

## HEXAGONAL GROUPS.

§ 1. Groups of this system have a series of parallel axes isomorphous with a 6-al (in the case of  $C_{3h}^m$  and  $D_{3h}^m$  with a 3-al) rotation-axis, these are called principal axes; planes perpendicular to them are (as before) called principal planes. There is only one possible translation-group, namely,  $\Gamma_h$ ; for  $\Gamma_{r,h}$  has not the symmetry of any point-group of the hexagonal system.

## § 2. TRIGONAL PARAMORPHY.

Since  $C_{3h}$  is derived from  $C_3$  by multiplying by a reflexion in a plane perpendicular to the 3-al axis, therefore groups  $C_{3h}^m$  are derived from those groups  $C_3^m$  which satisfy the condition of p. 165, and have  $\Gamma_h$  as their translation-group—that is, from  $C_3^1$  only—by multiplying by an operation  $S(t)$  in a principal plane which brings the system of axes of  $C_3^1$  into self-coincidence. To satisfy this last condition  $t$  (which must be half a translation in a principal plane) must be 0; for no two axes of  $C_3^1$  are at a distance from one another equal to half a primitive translation. Hence we have only a single group of this class

$$C_{3h}^1 = \{C_3^1, S_h\}. \quad \text{Subgroup } C_3^1.$$

It has as its symmetry-planes a set of principal planes of interval  $\tau_s$ .

## § 3. TRIGONAL HOLOHEDRY.

Exactly in the same way it is seen that groups  $D_{3h}^m$  can only be derived by multiplying  $D_3^1$  and  $D_3^2$  by a reflexion in a principal plane  $\sigma$ . In this case, however, the position of  $\sigma$  is no longer arbitrary; it must either contain 2-al axes when it is denoted by  $\sigma_h$ , or lie halfway between them when it is denoted by  $\sigma_m$ .

We have then

$$\begin{array}{ll} D_{3h}^1 = \{D_3^1, S_h\}. & \text{Subgroups } C_{3h}^1; C_{3v}^1. \\ D_{3h}^2 = \{D_3^1, S_m\}. & \text{Subgroups } C_{3h}^1; C_{3v}^3. \\ D_{3h}^3 = \{D_3^2, S_h\}. & \text{Subgroups } C_{3h}^1; C_{3v}^2. \\ D_{3h}^4 = \{D_3^2, S_m\}. & \text{Subgroups } C_{3h}^1; C_{3v}^4. \end{array}$$

#### § 4. HEXAGONAL TETARTOEDRY.

Groups of this class must contain a series of parallel screw-axes of angles  $\frac{\pi}{3}$ , whose translations have one of the values  $\frac{m\tau_z}{3}$  ( $m = 0, 1, 2, 3, 4, 5$ ). The translations of all reduced screws of the same angle must be equal, for  $\Gamma_h$  has no translation whose component parallel to the axis is  $< 2\tau_z$  (p. 161).

Using the notation of Fig. 154, let the axis  $e$  of a screw  $A_{\frac{\pi}{3}, t}$  pass through  $E$ ; then, transforming  $e$  by the translations, we see that the groups of this class have a series of similar screw-axes passing through the points  $E, E_1, E_2, E_3, \dots$ . (This is consistent with the construction of Fig. 80, p. 148).

Again, they have a series of screw-axes of angle  $\frac{2\pi}{3}$  and translation  $2t$  through points such as  $F$  and  $G$ , since  $A_{\frac{\pi}{3}, t}$  about  $e$ , followed by a similar screw about  $e_1$ , is equivalent, by Euler's construction, to a screw  $B_{\frac{2\pi}{3}, 2t}$  about  $f$ . They have also a series of screw-axes of angle  $\frac{\pi}{3}$  and translation  $3t$  lying halfway between any two screw-axes similar to  $e$ , for  $A_{\frac{\pi}{3}, t}$  about  $e$ , followed by  $B_{\frac{2\pi}{3}, 2t}$  about  $f$ , is equivalent to a screw  $D_{\frac{\pi}{3}, 3t}$  about an axis  $h$  halfway between  $E$  and  $E_1$ .

There are evidently six groups of this class

$$C_6^1 = \{A_{\frac{\pi}{3}, 0}, \Gamma_h\}.$$

$$C_6^2 = \{A_{\frac{\pi}{3}, \frac{\tau_z}{3}}, \Gamma_h\}.$$

$$C_6^3 = \{A_{\frac{\pi}{3}, \frac{2\tau_z}{3}}, \Gamma_h\}.$$

$$C_6^4 = \{A_{\frac{\pi}{3}, \frac{3\tau_z}{3}}, \Gamma_h\}.$$

$$C_6^5 = \{A_{\frac{\pi}{3}, \frac{4\tau_z}{3}}, \Gamma_h\}.$$

$$C_6^6 = \{A_{\frac{\pi}{3}, \tau_z}, \Gamma_h\}.$$

The arrangements of the axes of  $C_6^1$ ,  $C_6^2$ ,  $C_6^4$ , and  $C_6^5$  are shown in Figs. 172 to 175\*; the arrangements of  $C_6^3$  and  $C_6^5$  only differ from those of  $C_6^2$  and  $C_6^4$  respectively, in having right-handed screws instead of left-handed and vice versa.

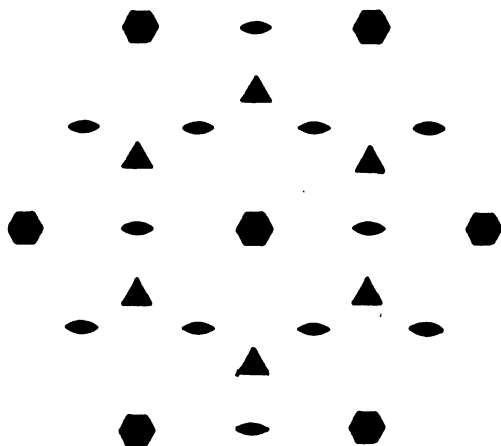


Fig. 172.  $C_6^1$ .

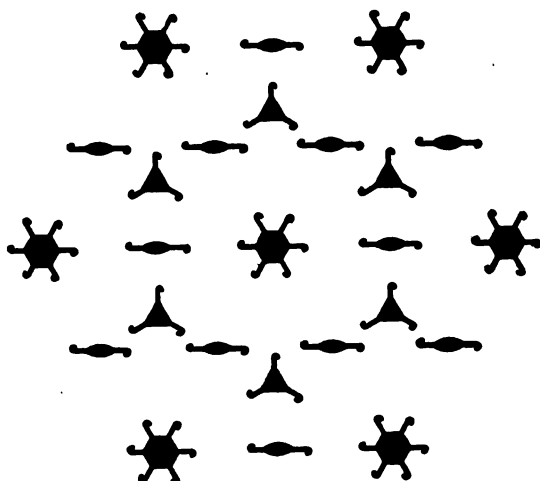




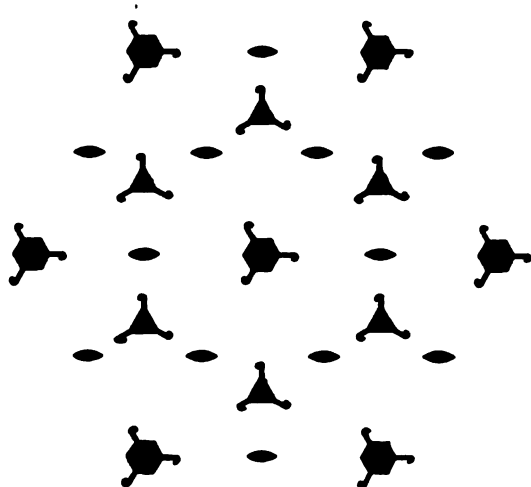
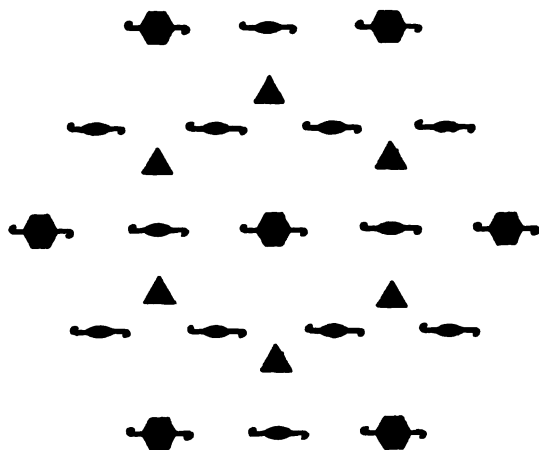


Fig. 173.  $C_6^2$ .

\* , , , , represent screw-axes of angle  $\frac{\pi}{8}$  and translations  $0$ ,  $\frac{\tau_2}{8}$ ,  $\frac{3\tau_2}{8}$ ,  $\tau_2$  respectively.

Fig. 17a.  $C_6^4$ .Fig. 17b.  $C_6^2$ .

## § 5. HEXAGONAL PARAMORPHY.

Since  $C_{6h}$  is derived from  $C_6$  by multiplying by an inversion, therefore the groups  $C_{6h}^m$  are derived from those groups  $C_6^m$  which satisfy the condition of p. 165—that is,  $C_6^1$  and  $C_6^5$ —by multiplying by an inversion (about a point  $P$ ) which brings the system of axes to self-coincidence. To fulfil this condition  $P$  must be such a point as  $E$  or the middle of  $EE_4$  (Fig. 154). Since, however, an inversion about the middle point of  $EE_4$

followed by the translation  $E_4E$  is equivalent to an inversion about  $E$ , it is sufficient to take  $P$  at  $E$ .

We have then

$$C_{6h}^1 = \{C_6^1, I\}. \quad \text{Subgroup } C_6^1.$$

$$C_{6h}^2 = \{C_6^2, I\}. \quad \text{Subgroup } C_6^2.$$

Both groups have for symmetry-planes a set of principal planes of interval  $\tau_z$ . The centres of symmetry lie in these planes in the case of  $C_{6h}^1$ , and halfway between these planes in the case of  $C_{6h}^2$ .

### § 6. HEXAGONAL HEMIMORPHY.

Exactly as in § 9 of chapter xx it may be shown that all groups  $C_{6v}^m$  are obtained by combining  $C_6^1$  or  $C_6^2$  with an operation  $S$  or  $S(\tau_z)$  in a plane  $\sigma$  parallel to the axes which brings the system of axes to self-coincidence. If this last condition is fulfilled  $\sigma$  must pass through such a line as  $EE_4$  or  $EF$  of Fig. 154 (cf. § 5 of chap. xxi).

We have then (Figs. 176 to 179):—

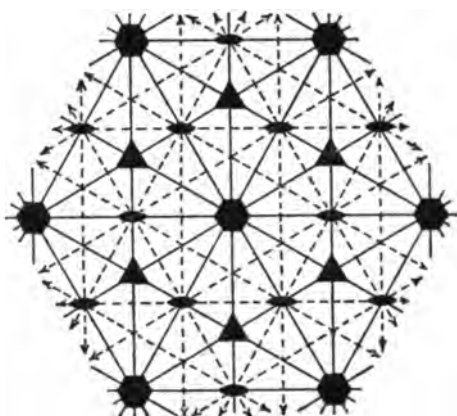


Fig. 176.  $C_{6v}^1$ .

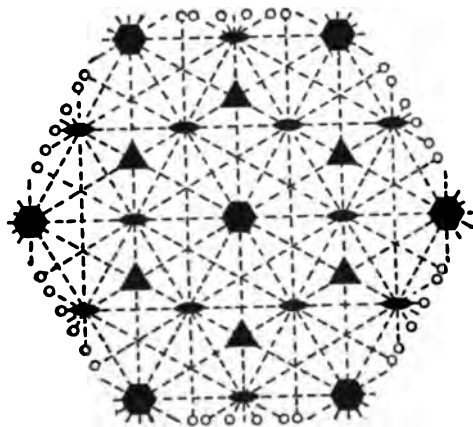
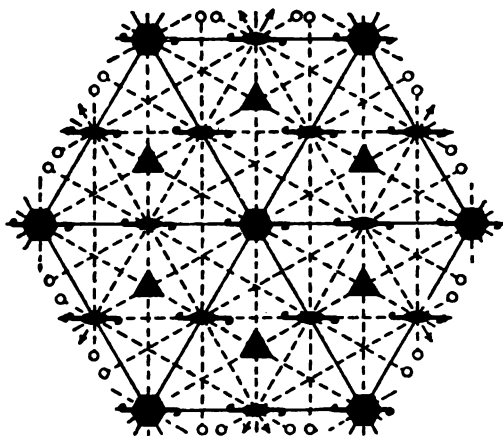
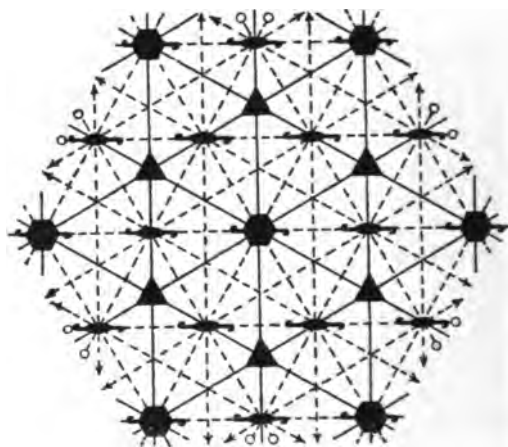


Fig. 177.  $C_{6v}^2$ .

Fig. 178.  $C_{6v}^3$ .Fig. 179.  $C_{6v}^4$ .

$C_{6v}^1 = \{C_6^1, S_6\} = \{C_6^1, S_6\}$ .  
Subgroups  $C_3^1(6)$ .

$C_{6v}^2 = \{C_6^1, S_6(\tau_z)\} = \{C_6^1, S_6(\tau_z)\}$ .  
Subgroups  $C_2^1(6)$ .

$C_{6v}^3 = \{C_6^2, S_6\} = \{C_6^2, S_6(\tau_z)\}$ .  
Subgroups  $C_3^2(3)$ ;  $C_2^2(3)$ .

$C_{6v}^4 = \{C_6^2, S_6(\tau_z)\} = \{C_6^2, S_6\}$ .  
Subgroups  $C_2^2(3)$ ;  $C_3^3(3)$ .

## § 7. HEXAGONAL ENANTIOMORPHY.

Exactly as in the case of  $D_3^m$  (p. 220) it may be shown that the groups  $D_6^m$  are derived from  $C_6^m$  by multiplying by a rotation about a line such as  $EE_4$  or  $EF$ .

Since, however, the screw of angle  $\frac{\pi}{3}$  about  $e$ , followed by a rotation through  $\pi$  about  $EE_4$ , is equivalent to a rotation through  $\pi$  about a line parallel to  $EF$ , the groups  $\{C_6^m, U_s\}$  and  $\{C_6^m, U_a\}$  are identical.

We have then

$$D_6^1 = \{C_6^1, U_a\}.$$

$$D_6^2 = \{C_6^2, U_a\}.$$

$$D_6^3 = \{C_6^3, U_a\}.$$

$$D_6^4 = \{C_6^4, U_a\}.$$

$$D_6^5 = \{C_6^5, U_a\}.$$

$$D_6^6 = \{C_6^6, U_a\}.$$

$D_6^1$  has axes arranged as in Fig. 180 in each member of a set of principal planes of interval  $\tau_z$ .

For the other groups the arrangement is similar, but the axes of the various subgroups  $C_3^m$  do not all intersect.

In the case of  $D_6^3$  axes parallel to  $EE_4$  lie in each member of a set of principal planes of interval  $\tau_z$ , while axes parallel to  $EG$ ,  $EE_2$ ,  $EF'$ ,  $EE_5$ ,  $EG'$  lie in the planes obtained by transforming this set

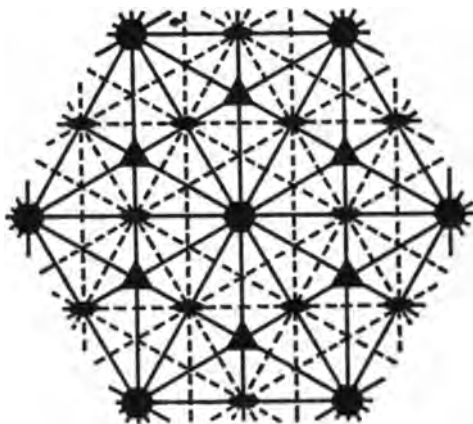


Fig. 180.  $D_6^1$ .

by  $\frac{\tau_z}{6}$ ,  $\frac{\tau_z}{3}$ ,  $\frac{\tau_z}{2}$ ,  $\frac{2\tau_z}{3}$ ,  $\frac{5\tau_z}{6}$  respectively.

The same holds for  $D_6^3$  if we put  $-\tau_z$  for  $\tau_z$ .

In the case of  $D_6^4$  axes parallel to  $EE_4$  and  $EF'$  lie in each member of a set of principal planes of interval  $\tau_z$ , while axes parallel to  $EG$  and  $EE_5$  lie in the planes obtained by trans-

forming this set by  $\frac{\tau_s}{3}$ , and axes parallel to  $EE_2$  and  $EG'$  lie in the planes obtained by transforming the set by  $\frac{2\tau_s}{3}$ .

The same holds for  $D_6^5$  if we put  $-\tau_s$  for  $\tau_s$ .

In the case of  $D_6^6$  axes parallel to  $EE_1$ ,  $EE_2$ , and  $EE_3$  lie in each member of a set of principal planes of interval  $\tau_s$ , and axes parallel to  $EG$ ,  $EF''$ , and  $EG'$  lie in the planes halfway between any two consecutive members of this set.

### § 8. HEXAGONAL HOLOHEDRY.

Groups  $D_{6h}^m$  are derived from  $D_6^1$  and  $D_6^6$ , just as groups  $C_{6h}^m$  were derived from  $C_6^1$  and  $C_6^6$ , by multiplying by an inversion about a point  $P$ , which may be taken as lying in a 6-al axis. If  $P$  lies at a point where a 6-al axis of  $D_6^1$  is met by a perpendicular rotation-axis or halfway between two points where it is so met, inversion about  $P$  evidently brings the system of axes of that group to self-coincidence, and otherwise not. Again if  $P$  lies at a point where a 6-al screw-axis of  $D_6^6$  is met by a perpendicular rotation-axis parallel to  $EF$ , or by a perpendicular rotation-axis parallel to  $EE_1$ , inversion about  $P$  evidently brings the system of axes of that group to self-coincidence and otherwise not.

Hence we have

$D_{6h}^1 = \{D_6^1, I\}.$	Subgroups $C_{6h}^1; C_{6v}^1.$
$D_{6h}^2 = \{D_6^1, I\}.$	Subgroups $C_{6h}^2; C_{6v}^2.$
$D_{6h}^3 = \{D_6^6, I_\alpha\}.$	Subgroups $C_{6h}^3; C_{6v}^3.$
$D_{6h}^4 = \{D_6^6, I_\beta\}.$	Subgroups $C_{6h}^4; C_{6v}^4.$

## CHAPTER XXIII

## REGULAR GROUPS.

§ 1. All groups of the regular system have one of the three translation-groups  $\Gamma_r, \Gamma_r', \Gamma_r''$ , whose primitive triplets may be taken as  $2\tau_x, 2\tau_y, 2\tau_z; \tau_y + \tau_z, \tau_z + \tau_x, \tau_x + \tau_y$ ;

and  $\tau_y + \tau_z - \tau_x, \tau_z + \tau_x - \tau_y, \tau_x + \tau_y - \tau_z$ ; respectively.

In each case  $\tau_x, \tau_y, \tau_z$  are equal in magnitude.

## § 2. REGULAR TETARTOEDRY.

Since the point-group  $T$  has a subgroup  $Q$  and four subgroups  $C_3$ , whose axes make equal angles with the axes of  $Q$ , therefore any group  $T^m$  has a subgroup  $Q^m$  and four subgroups  $C_3^m$ , whose axes make equal angles with the axes of  $Q^m$ .

The subgroup  $Q^m$  is specialized (by the fact that its orthorhombic translation-group is specialized to a regular translation-group) in such a way that the parallelepipedon  $p$  of Figs. 117 to 125 (p. 192) is a cube. The axes of the subgroups  $C_3^m$  run parallel to the diagonals of this cube; and it is at once evident from the position of these diagonals with regard to the translations, that these subgroups can only be  $C_3^4$ , and that hence

they all contain rotation-axes. Rotation through  $\frac{2\pi}{3}$  about

any one of these axes brings the system of axes of  $Q^m$  into self-coincidence; hence the arrangement of each of the three series of axes of  $Q^m$  must be the same; therefore  $Q^m$  must be one of  $Q^1, Q^4, Q^7, Q^8$ , or  $Q^9$ . The group  $T^m$  may be derived from its subgroup  $Q^m$  by multiplying by a rotation  $A$  through

$\frac{2\pi}{3}$  about any one of its 3-al rotation-axes  $a$ , for  $T$  can be derived from its subgroup  $Q$  by multiplying by a similar rotation about one of its 3-al axes.

Since a rotation through  $\frac{2\pi}{3}$  about  $a$  brings the system of axes to self-coincidence,  $a$  must evidently be some diagonal of  $p$ . These diagonals have all the same position with regard to the system of axes as a whole (unless the subgroup is  $Q^7$ ,

when  $a$  must coincide with  $AA_1$  of Fig. 126). We hence have the same group  $T^m$  whichever diagonal we take for  $a$ ; we shall take  $AA_1$ . We have then

$$T^1 = \{Q^1, A\}.$$

$$T^2 = \{Q^2, A\}.$$

$$T^3 = \{Q^3, A\}.$$

$$T^4 = \{Q^4, A\}.$$

$$T^5 = \{Q^5, A\}.$$

The arrangements of the axes of the regular groups are too complicated to be shown readily in diagrams; but the following considerations will make it clear in the present case. The arrangement of the axes of each subgroup  $C_3^m$  is the same, and so is that of each subgroup  $C_3^4$ ; the arrangement of the 3-al axes of any one of the latter is given by Figs. 158 and 159 when the translation-group is known\* ( $\Gamma_r$  for  $T^1$  and  $T^4$ ,  $\Gamma_r'$  for  $T^2$ , and  $\Gamma_r''$  for  $T^3$  and  $T^5$ ), if the position of one rotation-axis is also known. This is found in the case of  $T^1$ ,  $T^2$ , and  $T^3$  by noticing that if a 2-al and a 3-al rotation-axis intersect at an angle  $\cos^{-1} \frac{1}{\sqrt{3}}$ , three other 3-al rotation-axes pass

through their intersection (cf. p. 60), so that rotation-axes of each subgroup  $C_3^4$  of  $T^1$ ,  $T^2$ , and  $T^3$  pass through the vertex  $A$  of  $p$  (Fig. 181). In  $T^4$  and  $T^5$  no two 3-al rotation-axes can meet; the relative arrangement is easily found by transforming the axis  $a$  by the various screws of angle  $\pi$ , and is shown in Fig. 182.

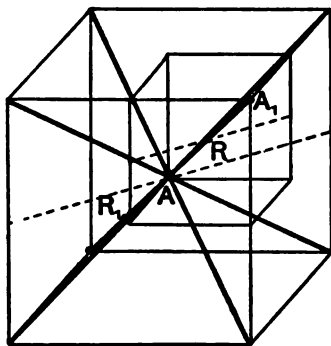


Fig. 181.

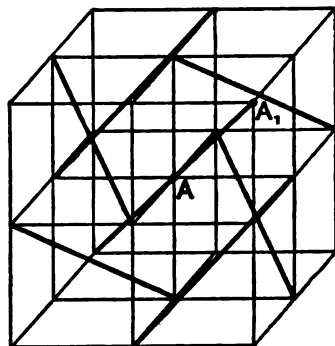


Fig. 182.

\* In § 1 the primitive triplets were put in a form the same as that of p. 141; that is, each triplet consists of three equal translations all equally inclined to the 3-al axes, and each equally inclined to the other two.

### § 3. REGULAR PARAMORPHY.

Since  $T_h$  is derived from  $T$  by multiplying by an inversion, the groups  $T_h^m$  are derived from  $T^m$  by multiplying by an inversion\* (about a point  $P$ ) which brings the system of axes to self-coincidence.

From p. 166 we know that if the inversion brings the system of axes of the subgroup  $Q^m$  and of one of the subgroups  $C_3^4$  to self-coincidence it brings the whole system of  $T^m$  to self-coincidence, and thus forms a group  $T_h^m$  from  $T^m$ . Now by combining the operations of  $C_3^4$  with an inversion we get  $C_{3i}^4$ , and this group has a centre of symmetry in each rotation-axis.

Hence it is sufficient to take  $P$  in any given 3-al rotation-axis of  $T^m$ ; we shall suppose  $P$  lies in the diagonal  $AA_1$  of the cube  $p$  (Fig. 126), which is always a 3-al rotation-axis of  $T^m$ .

If, moreover, the inversion is to bring the system of axes of the subgroup  $Q^m$  to self-coincidence,  $P$  must be at a vertex or at the centre of  $p$ ; that is, at  $A$  or  $R$  of Fig. 126. We see, however, from pp. 195, 197, that the groups obtained from  $T^3$ ,  $T^4$ , and  $T^5$  (whose subgroups are  $Q^4$ ,  $Q^3$ , and  $Q^5$  respectively) by giving  $P$  these two positions in turn are not distinct. The arrangement of the axes and planes of  $T_h^m$  is given completely when we know the subgroups  $T^m$  and  $Q_h^m$ .

$T_h^1 = \{T^1, I\}.$	Subgroup $Q_h^1.$
$T_h^2 = \{T^1, I_r\}.$	Subgroup $Q_h^2.$
$T_h^3 = \{T^2, I\}.$	Subgroup $Q_h^{23}.$
$T_h^4 = \{T^2, I_r\}.$	Subgroup $Q_h^{24}.$
$T_h^5 = \{T^3, I\}.$	Subgroup $Q_h^{25}.$
$T_h^6 = \{T^4, I\}.$	Subgroup $Q_h^{16}.$
$T_h^7 = \{T^5, I\}.$	Subgroup $Q_h^{27}.$

### § 4. REGULAR HEMIMORPHY.

Since  $T_d$  is derived from  $T$  by multiplying by a reflexion in a plane through a 3-al and a 2-al axis, therefore the groups  $T_d^m$  are derived from  $T^m$  by multiplying by an operation  $S(t)$  in a plane parallel to the axes of any two subgroups  $C_2^m$  and  $C_3^4$  which brings the system of axes of  $T^m$  to self-coincidence. This last condition is fulfilled (p. 166) if  $S(t)$  brings the system of the subgroup  $Q^m$  and that of one subgroup  $C_3^4$  to

\* All groups  $T^m$  satisfy the condition of p. 165.

self-coincidence. We may take the plane of  $S(t)$  parallel to  $NRFA$  (Fig. 126).

Every subgroup  $C_{3v}^m$  of  $T_d^m$  is either  $C_{3v}^5$  or  $C_{3v}^6$ ; for every subgroup  $C_3^m$  of  $T^m$  is  $C_3^4$ . Therefore every group  $T_d^m$  has a gliding-plane passing through a 3-al rotation-axis, whose translation is either zero or half the primitive translation parallel to that axis. Hence, in order to obtain all groups of the form  $\{T^m, S(t)\}$ , it is sufficient to take  $NRFA$  as the plane of  $S(t)$ , and to take  $t = 0$  or  $=$  half a translation in the direction  $AA_1$ .

$S(t)$  forms with  $Q^m$  a group  $D_{2d}^m$ ; since no group of this class can be formed from  $Q^4$  (p. 199), no group  $T_d^m$  can be formed from  $T^4$ . No group  $D_{2d}^m$  can be formed from  $Q^8$  by an operation  $S(t)$  in the plane  $NRFA$  if  $t = \frac{1}{2}(\tau_x + \tau_y + \tau_z)$ , nor from  $Q^9$  if  $t = 0$ . Hence we have

$$T_d^1 = \{T^1, S_d\}. \quad \text{Subgroup } D_{2d}^1.$$

$$T_d^2 = \{T^2, S_d\}. \quad \text{Subgroup } D_{2d}^2.$$

$$T_d^3 = \{T^3, S_d\}. \quad \text{Subgroup } D_{2d}^3.$$

These three groups have each four subgroups  $C_{3v}^5$ .

$$T_d^4 = \{T^1, S_d(\tau_x + \tau_y + \tau_z)\}. \quad \text{Subgroup } D_{2d}^2.$$

$$T_d^5 = \{T^2, S_d(\tau_x + \tau_y + \tau_z)\}^*. \quad \text{Subgroup } D_{2d}^{10}.$$

$$T_d^6 = \left\{T^3, S_d\left(\frac{\tau_x + \tau_y + \tau_z}{2}\right)\right\}. \quad \text{Subgroup } D_{2d}^{12}.$$

These three groups have each four subgroups  $C_{3v}^6$ .

### § 5. REGULAR ENANTIOMORPHY.

Since  $O$  may be derived from  $T$  by a rotation through  $\pi$  about any axis perpendicular to a 3-al and a 2-al axis, the groups  $O^m$  may be derived from  $T^m$  by multiplying by an operation  $U_{\pi, u}$  about an axis  $u$  perpendicular to the axes of any two subgroups  $C_3^4$  and  $C_2^m$  which brings the system of axes to self-coincidence. We may take  $u$  parallel to  $LM$  (Fig. 126).

Since  $U$  forms with a subgroup  $C_3^4$  of  $T^m$  a subgroup  $D_3^7$  of  $O^m$ —for  $D_3^7$  is the only group derivable from  $C_3^4$  by an operation such as  $U$ —and since  $D_3^7$  has a 2-al rotation-axis meeting each 3-al rotation-axis, therefore it is sufficient to

\* This is equivalent to  $\{T^2, S_d(\tau_z)\}$  since  $\tau_x + \tau_y$  is a translation.

consider the cases in which the translation  $t$  of  $U$  is equal to zero and in which  $u$  intersects  $AA_1$ .

Of course two groups are identical if they are derived from the same group  $T^m$  by rotations through  $\pi$  about two lines which are both rotation-axes in the same subgroup  $D_3^7$ .

In the case of  $T^1$   $u$  must evidently pass through a point such as  $A$  or  $R$  of Fig. 126 (cf. Fig. 181), for a rotation through  $\pi$  about  $u$  brings the system of axes to self-coincidence.

$$O^1 = \{T^1, U\}. \quad \text{Subgroups } D_4^1 (3) *.$$

$$O^2 = \{T^1, U_r\}. \quad \text{Subgroups } D_4^5 (3).$$

Similarly in the case of  $T^2$  we have

$$O^3 = \{T^2, U\}. \quad \text{Subgroups } D_4^9 (3).$$

$$O^4 = \{T^2, U_r\}. \quad \text{Subgroups } D_4^{10} (3).$$

Since  $T^2$  has a translation  $\tau_x + \tau_y + \tau_z$ , the groups  $\{T^2, U\}$  and  $\{T^2, U_r\}$  are identical.

$$O^5 = \{T^2, U\}. \quad \text{Subgroups } D_4^9 (3).$$

In the case of  $T^4$   $u$  must evidently pass through the middle point of  $A_1R$  or the middle point of  $RA$  (see Fig. 183), or through similar points.

$$O^6 = \{T^4, U_1\}. \\ \text{Subgroups } D_4^4 (3).$$

$$O^7 = \{T^4, U_2\}. \\ \text{Subgroups } D_4^8 (3).$$

These two groups are alike except that where one has left-handed screws the other has right-handed, and vice versa.

For groups derived from  $T^6$  the axis  $u$  must have a position similar to that in the case of groups derived from  $T^4$ . Since, however, the group  $T^6$  has a translation  $\tau_x + \tau_y + \tau_z$ , the groups  $\{T^6, U_1\}$  and  $\{T^6, U_2\}$  are identical.

$$O^8 = \{T^6, U\}. \quad \text{Subgroups } D_4^{10} (3).$$

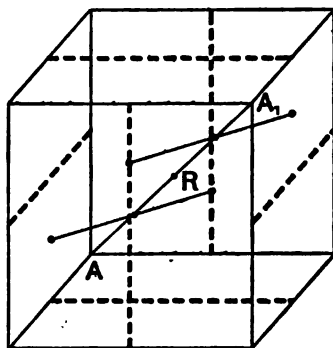


Fig. 183.

\* That is, the group has three subgroups  $D_4^*$  each of which is  $D_4^1$ . Each group  $O^*$  has also four subgroups  $D_3^7$ .

## § 6. REGULAR HOLOHEDRY.

Since  $O_h$  is derived from  $O$  by multiplying by an inversion, therefore groups  $O_h^m$  are derived from those groups  $O^m$  which satisfy the condition of p. 165—that is,  $O^1$ ,  $O^2$ ,  $O^3$ ,  $O^4$ ,  $O^5$ , and  $O^6$ —by multiplying by an inversion (about a point  $P$ ) which brings the system of axes to self-coincidence. This last condition is fulfilled if and only if the inversion brings to self-coincidence the system of axes of the subgroup  $T^m$  and of that subgroup  $C_2^m$  whose axes are parallel to the line  $u$  of the previous section (p. 166).

Hence it is necessary and sufficient to consider those cases in which  $P$  lies at a vertex or at the centre of the cube  $p^*$ , and also on an axis such as  $u$ , or halfway between two rotation-axes parallel to  $u$ .

We have then readily

$$O_h^1 = \{O^1, I\}. \quad \text{Subgroups } T_h^1; D_{3d}^1(4); D_{4h}^1(3).$$

$$O_h^2 = \{O^1, I_r\}. \quad \text{Subgroups } T_h^2; D_{3d}^2(4); D_{4h}^2(3).$$

$$O_h^3 = \{O^2, I\}. \quad \text{Subgroups } T_h^1; D_{3d}^2(4); D_{4h}^2(3).$$

$$O_h^4 = \{O^2, I_r\}. \quad \text{Subgroups } T_h^2; D_{3d}^2(4); D_{4h}^2(3).$$

From  $O^1$  and  $O^2$  we get the same group  $O_h^m$  if we multiply by an inversion about  $A_1$  as if we multiply by an inversion about  $A$ , for  $AA_1$  is half a translation of  $O^1$  and  $O^2$ . This is not the case for  $O^3$ ; we have in fact

$$O_h^5 = \{O^3, I\}. \quad \text{Subgroups } T_h^3; D_{3d}^3(4); D_{4h}^3(3).$$

$$O_h^6 = \{O^3, I_1\}. \quad \text{Subgroups } T_h^3; D_{3d}^3(4); D_{4h}^3(3).$$

There can be no group  $\{O^3, I_r\}$ , for the centre of  $p$  is not on or halfway between axes parallel to  $u$ . For a similar reason there can be no groups  $\{O^4, I\}$  or  $\{O^4, I_1\}$ , but there are two groups derivable from  $O^4$  by taking  $P$  either halfway between  $A$  and  $A_1$  or at the same distance from  $A$  on the line  $AA_1$  but on the opposite side of  $A$  ( $R_1$  of Fig. 181).

$$O_h^7 = \{O^4, I_r\}. \quad \text{Subgroups } T_h^4; D_{3d}^4(4); D_{4h}^4(3).$$

$$O_h^8 = \{O^4, I_{r_1}\}. \quad \text{Subgroups } T_h^4; D_{3d}^4(4); D_{4h}^4(3).$$

$O^5$  and  $O^6$  have a translation  $\tau_x + \tau_y + \tau_z$ ; therefore  $\{O^5, I\}$

\* With an exception in the case of  $O_h^2$ .

and  $\{O^6, I_r\}$  on the one hand and  $\{O^8, I\}$  and  $\{O^8, I_r\}$  on the other are identical.

$O_h^9 = \{O^5, I\}$ . Subgroups  $T_h^5$ ;  $D_{3d}^5(4)$ ;  $D_{4h}^{17}(3)$ .

$O_h^{10} = \{O^8, I\}$ . Subgroups  $T_h^7$ ;  $D_{3d}^8(4)$ ;  $D_{4h}^{20}(3)$ .

§ 7. The following table shows the distribution of the 230 possible groups of movements which we have now obtained:—

TRICLINIC SYSTEM		$\Gamma_{tr}$			
$C_1$	Hemihedry . . . . .	1	. . . . .	1	} 2.
$C_i$	Holohedry . . . . .	1	. . . . .	1	
MONOCLINIC SYSTEM		$\Gamma_m, \Gamma_m'$			
$C_2$	Hemihedry . . . . .	2	2 . . . . .	4	} 13.
$C_2$	Hemimorphy . . . . .	2	1 . . . . .	3	
$C_{2h}$	Holohedry . . . . .	4	2 . . . . .	6	
ORTHORHOMBIC SYSTEM		$\Gamma_o, \Gamma_o', \Gamma_o'', \Gamma_o'''$			
$C_{2v}$	Hemimorphy . . . . .	10	7 2 3 . . . . .	22	} 59.
$Q$	Enantiomorphy . . . . .	4	2 1 2 . . . . .	9	
$Q_h$	Holohedry . . . . .	16	6 2 4 . . . . .	28	
TETRAGONAL SYSTEM		$\Gamma_t, \Gamma_t'$			
$C_4'$	Tetartohedry of the second sort . . . . .	1	1 . . . . .	2	} 68.
$D_{2d}$	Hemihedry of the second sort . . . . .	8	4 . . . . .	12	
$C_4$	Tetartohedry . . . . .	4	2 . . . . .	6	
$C_{4h}$	Paramorphy . . . . .	4	2 . . . . .	6	
$C_{4v}$	Hemimorphy . . . . .	8	4 . . . . .	12	
$D_4$	Enantiomorphy . . . . .	8	2 . . . . .	10	
$D_{4h}$	Holohedry . . . . .	16	4 . . . . .	20	
RHOMBOHEDRAL SYSTEM		$\Gamma_h, \Gamma_{rh}$			
$C_3$	Tetartohedry . . . . .	3	1 . . . . .	4	} 25.
$C_{3i}$	Hexagonal tetartohedry of the second sort . . . . .	1	1 . . . . .	2	
$C_{3v}$	Hemimorphy . . . . .	4	2 . . . . .	6	
$D_3$	Enantiomorphy . . . . .	6	1 . . . . .	7	
$D_{3d}$	Holohedry . . . . .	4	2 . . . . .	6	

HEXAGONAL SYSTEM		$\Gamma_h$	
$C_{3h}$	Trigonal paramorphy . . . . .	1	1
$D_{3h}$	Trigonal holohedry . . . . .	4	4
$C_6$	Tetartohedry . . . . .	6	6
$C_{6h}$	Paramorphy . . . . .	2	2
$C_{6v}$	Hemimorphy . . . . .	4	4
$D_6$	Enantiomorphy . . . . .	6	6
$D_{6h}$	Holohedry . . . . .	4	4
		} 27.	

REGULAR SYSTEM		$\Gamma_r, \Gamma_r', \Gamma_r''$	
$T$	Tetartohedry . . . . .	2 1 2	5
$T_h$	Paramorphy . . . . .	3 2 2	7
$T_d$	Hemimorphy . . . . .	2 2 2	6
$O$	Enantiomorphy . . . . .	4 2 2	8
$O_h$	Holohedry . . . . .	4 4 2	10
		} 36.	

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## CHAPTER XXIV

## SPACE-PARTITIONING.

§ 1. We defined (p. 47) the series of points with which any given point is brought to coincidence by the various operations of any group  $\Gamma$  as a 'system of equivalent points,' and proved that in general a point is brought into coincidence with any given equivalent point by one and only one operation of  $\Gamma$ .

§ 2. We will now show how to fill up space with a series of cells each of which contains one point and one only of any equivalent system. Each such cell will be called a *fundamental cell*\*.

Take any number of points  $P, Q, R, \dots$  no two of which are equivalent, and draw closed surfaces of very small dimensions round each; now let each surface grow in all directions, the growth in any direction only stopping when further growth in that direction would include, within the surface, a point equivalent to some point already within one or other of the surfaces. When further growth is no longer possible without violating this condition let the portion of space included inside all the closed surfaces be  $\phi$ ; let  $\phi$  be brought to  $\phi, \phi_1, \phi_2, \phi_3, \dots$  by all the operations of any group of movements  $\Gamma$ . Any one of these portions is brought to coincidence with any other by some operation of  $\Gamma$ †. We shall show that  $\phi, \phi_1, \phi_2, \dots$  are fundamental cells. We start by proving that  $\phi, \phi_1, \phi_2, \dots$  fill space completely, and that no two of them overlap.

Firstly, they fill space completely; for, if not, suppose there is an interval of space  $A$  between  $\phi_i$  and  $\phi_m$  not included in  $\phi, \phi_1, \phi_2, \dots$ ; let  $L$  be the operation bringing  $\phi_i$  to coincide with  $\phi$ , and let  $L$  bring  $\phi_m$  to coincide with  $\phi_p$ . Then there is evidently an interval of space  $A'$  (with which  $A$  is brought to coincide by  $L$ ) between  $\phi$  and  $\phi_p$  not included in  $\phi, \phi_1, \phi_2, \dots$ ; and there is no point inside  $A'$  equivalent to any point

\* German, 'Fundamentalebene.' The cell must have closed boundaries, but not necessarily a *single* closed boundary.

† Proof as on p. 47.

inside  $\phi$ . Therefore  $\phi$  may be allowed to grow so as to absorb  $A'$  without violating the condition that  $\phi$  should contain no two equivalent points; but this would be contrary to the hypothesis that  $\phi$  cannot grow further without violating this condition.

Secondly, no two of  $\phi, \phi_1, \phi_2, \dots$  overlap. For if  $\phi_i$  and  $\phi_m$  overlap so must  $\phi$  and  $\phi_p$ . Let  $H$  be a point in the portion of space common to  $\phi$  and  $\phi_p$ ; let  $L'$  be the operation bringing  $\phi_p$  to coincide with  $\phi$ , and let  $L'$  bring  $H$  to coincide with  $K$ . Then since  $H$  is in  $\phi_p$ ,  $K$  must be in  $\phi$ ; but  $H$  is in  $\phi$ , and therefore  $\phi$  contains two equivalent points  $H$  and  $K$  contrary to hypothesis.

No one of the regions  $\phi, \phi_1, \phi_2, \dots$  contains in its interior two equivalent points. For if  $\phi_i$  contains two equivalent points  $P_i, P'_i$ , suppose the operation  $L$  of  $\Gamma$  brings  $\phi_i$  into coincidence with  $\phi$ , and  $P_i, P'_i$  to coincide with  $P, P'$ ; then  $P, P'$  are two equivalent points lying in  $\phi$  contrary to hypothesis.

Each one of  $\phi, \phi_1, \phi_2, \dots$  contains at least one of any system of equivalent points on its surface or in its interior. For let  $Q$  be any point of such a system; then  $Q$  must lie in or on one of  $\phi, \phi_1, \phi_2, \dots$ , for between them they fill all space; let  $Q$  lie in or on  $\phi_n$ . Then, since  $\phi_n$  can be brought to coincidence with *any one* of  $\phi, \phi_1, \phi_2, \dots$  by some operation of  $\Gamma$ , some point equivalent to  $Q$  must lie in or on each of  $\phi, \phi_1, \phi_2, \dots$ .

The only operation of  $\Gamma$  which can bring any one of the spaces  $\phi_v$  into coincidence with itself is the identical operation. For let  $L$  be an operation bringing  $\phi_v$  to self-coincidence, and let  $L$  bring any point  $P$  inside  $\phi_v$  to  $P'$ ; then  $P'$  is evidently inside  $\phi_v$ , and therefore  $\phi_v$  contains two equivalent points, which is impossible unless  $P$  and  $P'$  coincide, and hence  $L = 1$ .

Again  $\phi_v$  cannot be brought to coincide with the same space  $\phi_w$  by two distinct operations  $L$  and  $L'$  of  $\Gamma$ , for otherwise  $L' \cdot L^{-1}$  would be an operation of  $\Gamma$  distinct from unity bringing  $\phi_v$  to self-coincidence.

Now consider any point  $P$  on the boundary of  $\phi_n$ . Suppose  $P$  lies on the boundary between  $\phi_n$  and  $\phi_e$ . Let  $L$  be an operation bringing  $\phi_e$  to coincide with  $\phi_n$ ,  $\phi_n$  with  $\phi_f$ , and  $P$  with  $P'$ . Then obviously  $P'$  lies on the boundary between  $\phi_n$  and  $\phi_f$ , so that there is at least one point equivalent to  $P$  which is also on the boundary of  $\phi_n$ .

Conversely, there is *only* one such point, for let  $P''$  be any

point on the boundary of  $\phi_n$  equivalent to  $P$ , and let  $L'$  be the operation of  $\Gamma$  bringing  $P$  to coincide with  $P''$ .

Then, if  $P''$  lies on the boundary of  $\phi_n$  and  $\phi_g$ , evidently  $L'$  brings  $\phi_n$  to coincide with itself and  $\phi_g$  to coincide with  $\phi_g$ , or else it brings  $\phi_n$  to coincide with  $\phi_g$  and  $\phi_g$  to coincide with  $\phi_n$ . Now the first alternative is impossible unless  $L' = 1$ , and the second is impossible unless  $L' = L$ , that is, unless  $P''$  coincides with  $P'$ .

Finally, two equivalent points  $P$  and  $P'$  cannot be such that  $P$  lies on the surface of  $\phi_u$  and  $P'$  inside  $\phi_g$ ; for if  $L$  is the operation of  $\Gamma$  bringing  $P$  to coincide with  $P'$ , and if  $L$  brings  $\phi_u$  to coincide with  $\phi_u$ , then  $P'$  must lie on the surface of  $\phi_u$ , and cannot therefore lie inside  $\phi_g$ .

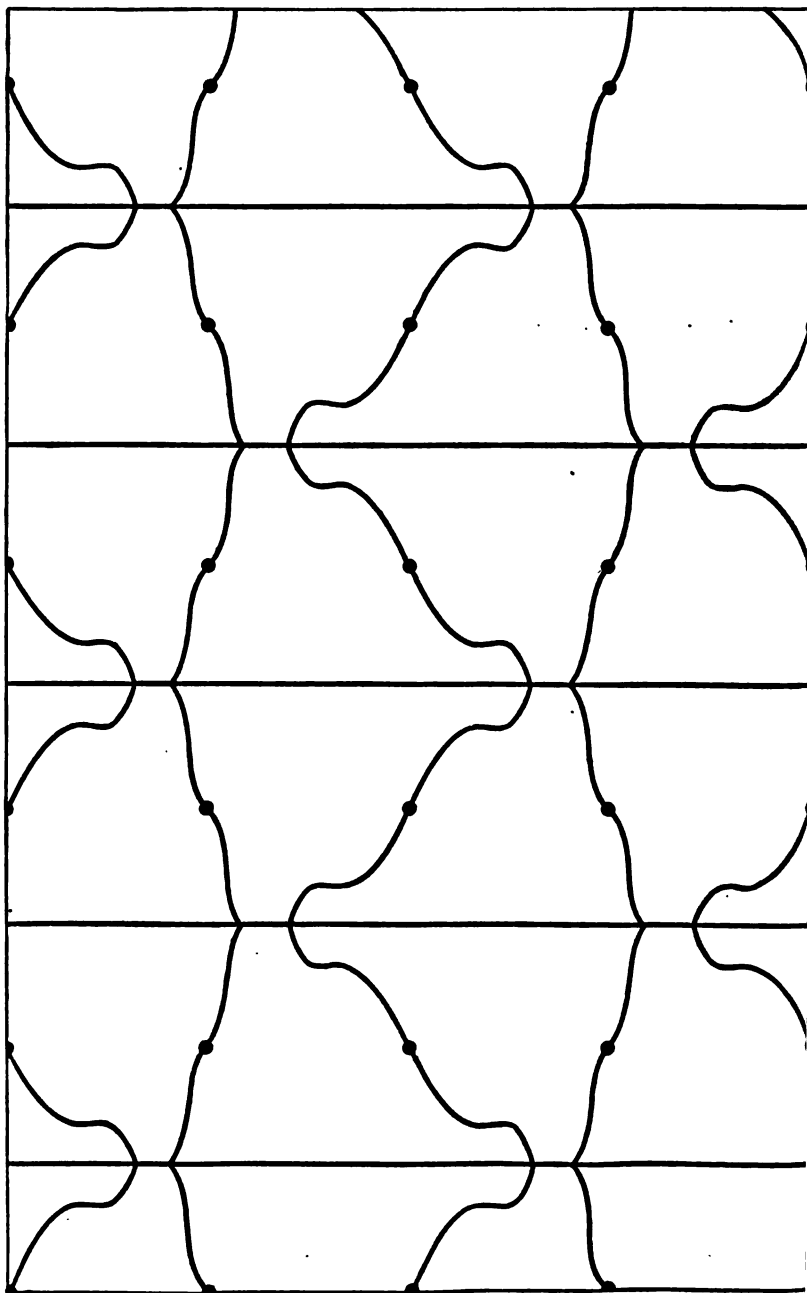
Hence we have proved that  $\phi, \phi_1, \phi_2, \dots$  are fundamental cells according to the definition of p. 241, if we consider a point lying on the surface between two of the series  $\phi, \phi_1, \phi_2, \dots$  as being half in one and half in the other.

§ 3. When the form and position of  $\phi$  are chosen the forms and positions of the cells  $\phi_1, \phi_2, \phi_3, \dots$  are fixed, for they are obtained by transforming  $\phi$  by the operations of  $\Gamma$ . They are all congruent to  $\phi$ , or half congruent and half enantiomorphous to  $\phi$ , according as  $\Gamma$  does not or does contain operations of the second sort. There is a considerable choice in obtaining the form and position of  $\phi$ . This cell cannot, however, be pierced by a rotation-axis or symmetry-plane; for if that were possible a point inside the cell sufficiently close to the axis or plane would be brought by rotation about the axis or reflexion in the plane to coincide with a point also inside  $\phi$ , so that  $\phi$  would contain two equivalent points contrary to hypothesis. Rotation-axes and symmetry-planes must therefore lie in the surfaces of fundamental cells.

Examples of two divisions of space into fundamental cells for the group  $C_{2v}^4$  are given in Figs. 184 and 185. The point of view is that of Fig. 95 (p. 180); that is, the observer is supposed to be looking in the direction of  $-\tau_z$ . The position of the 2-al axes is shown by dots; the position of the other elements of symmetry may be found from Fig. 95.

In Fig. 184 the cell has a single boundary which may be taken as a cylinder whose height is  $2\tau_z$ , and whose base is any one of the areas there shown.

In Fig. 185 each cell has three boundaries and may be taken as made up of three cylinders whose height is  $2\tau_z$ , and whose bases are any three areas labelled with the same number.



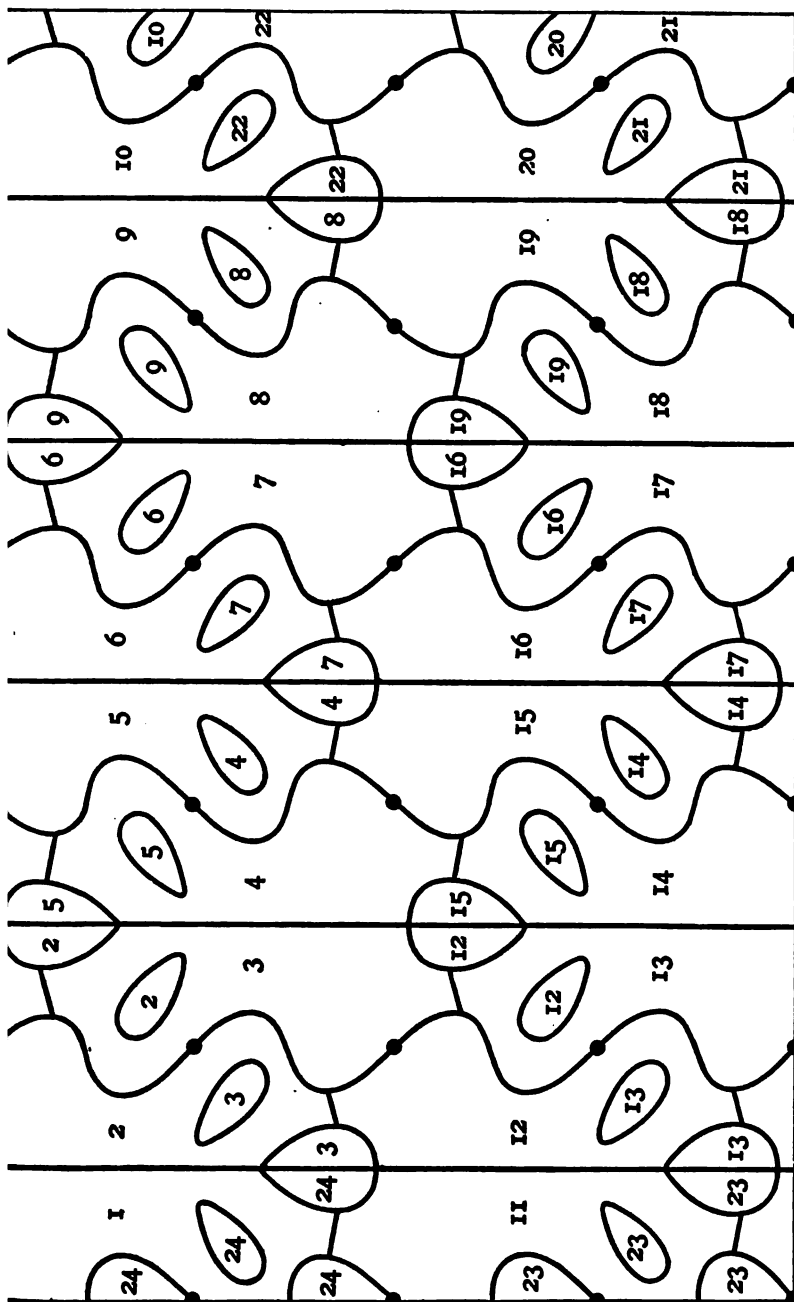


Fig 185.

The division of the cells into two enantiomorphous sets (due to the fact that  $C_{2v}^1$  has operations of the second sort) is obvious in the figures; for example, in Fig. 185, cells 2, 3, 6, 7, &c., form a set congruent to one another and all enantiomorphous to 1, 4, 5, 8, &c.

It will be noticed that the rotation-axes lie in the boundaries, and that the symmetry-planes (but not necessarily the gliding-planes) form part of the boundaries of the fundamental cells.

§ 4. We have split up space into portions  $\phi, \phi_1, \phi_2, \dots$  such that each one is brought to coincidence with some other by every operation of  $\Gamma$ , and such that no one is brought to coincide with itself by any operation of  $\Gamma$  except unity. It is sometimes possible to divide up space into portions  $\Phi, \Phi_1, \Phi_2, \dots$  each of which is brought into coincidence with itself or with some other by every operation of  $\Gamma$ , and is brought to coincidence with itself by operations of  $\Gamma$  other than unity.

Let  $1, L_1, L_2, L_3, \dots, L_m$  be the operations of  $\Gamma$  which bring  $\Phi$  to self-coincidence, then these operations must form a point-group for they all bring a *finite* figure to self-coincidence. Hence space cannot be divided into portions such as  $\Phi, \Phi_1, \Phi_2, \dots$  unless  $\Gamma$  has a point-group for a subgroup.

The converse is true that, if  $\Gamma$  has a point-group as a subgroup, regions such as  $\Phi, \Phi_1, \Phi_2, \dots$  can be found. For let the operations of a point-group, which is a subgroup of  $\Gamma$ , be  $1, L_1, L_2, L_3, \dots, L_m$ . . . . . (i), and let a fundamental cell  $\phi$  be brought to coincide with  $\phi, \phi_1, \phi_2, \dots, \phi_m$  by these operations. Now take an operation  $M$  of  $\Gamma$  not included in (i), and let  $M$  bring these cells to coincide with the cells  $\phi', \phi'_1, \phi'_2, \dots, \phi'_m$  respectively. Let the portion of space occupied by  $\phi, \phi_1, \phi_2, \dots, \phi_m$  taken together be denoted by  $\Phi$ , and that occupied by  $\phi', \phi'_1, \phi'_2, \dots, \phi'_m$  by  $\Phi_1$ . Now consider an operation  $R^*$  of  $\Gamma$  which brings  $\phi_e$  to coincide with  $\phi_f$ ; then  $L_e.R$  brings  $\phi$  to coincide with  $\phi_f$ , and therefore  $L_e.R = L_f$  or  $R = L_e^{-1}.L_f$ . Hence  $R$  brings  $\phi_u$  to coincide with that cell to which  $\phi$  is brought by  $L_u.R$ , that is, by  $L_u.L_e^{-1}.L_f = L_p$ , where  $L_p$  is one of the operations (i).

Therefore  $\phi_u$  is brought by  $R$  to coincidence with  $\phi_p$ , one of the cells composing  $\Phi$ ; and, since two different fundamental cells cannot be brought to coincide with the same

\* It is easily seen that  $R$  belongs to the series (i).

cell by the same operation,  $\Phi$  is brought to self-coincidence by every operation which brings one of the cells contained in it to coincidence with any other cell so contained.

Now consider an operation  $R_1$  of  $\Gamma$  which brings  $\phi_q$  to coincide with  $\phi'_j$ ; then, as before,  $R_1 = L_e^{-1} \cdot L_j \cdot M$  (for  $L_e^{-1} \cdot L_j$  brings  $\phi_e$  to coincide with  $\phi_j$  and  $M$  brings  $\phi_j$  to coincide with  $\phi'_j$ ). Hence  $R_1$  brings  $\phi_u$  to coincide with that cell with which  $\phi$  is brought to coincide by  $L_u \cdot L_e^{-1} \cdot L_j \cdot M = L_p \cdot M$ , i.e. with a cell  $\phi_p'$  contained in  $\Phi_1$ . Hence  $\Phi$  is brought to coincide with  $\Phi_1$  by every operation of  $\Gamma$  which brings one of the component cells of  $\Phi$  to coincide with one of the component cells of  $\Phi_1$ .

By taking other operations of  $\Gamma$  we can derive figures  $\Phi_2, \Phi_3, \dots$  and can fill all space with such figures. For if there is a fundamental cell  $\phi_n$  not included in  $\Phi, \Phi_1, \Phi_2, \Phi_3, \dots$ , we can always find an operation of  $\Gamma$  which brings  $\phi$  to coincide with  $\phi_n$ , derive a new figure from  $\Phi$  by its aid, add this figure to  $\Phi, \Phi_1, \Phi_2, \Phi_3, \dots$ , and repeat this process till all space is filled.

No two of these figures can overlap; for if  $\Phi_q$  and  $\Phi_r$  had a common cell  $\phi_o$ , and if  $R$  were the operation of  $\Gamma$  bringing  $\phi$  to coincide with  $\phi_o$ , then  $R$  would bring  $\Phi$  to coincide both with  $\Phi_q$  and  $\Phi_r$ , which is impossible.

Lastly, any figure  $\Phi_h$  is brought into coincidence with a similar figure by any operation  $E$  of  $\Gamma$ . For if  $\phi$  is brought to coincide with any cell contained in  $\Phi_h$  by the operation  $F$  of  $\Gamma$ ,  $\Phi$  is brought to coincide with  $\Phi_h$  by  $F$ . But  $F \cdot E$  is an operation of  $\Gamma$ ; hence  $\Phi$  is brought by  $F \cdot E$ , and therefore  $\Phi_h$  by  $F$ , into coincidence with itself or some other figure of the series  $\Phi, \Phi_1, \Phi_2, \dots$ .

This series therefore fills all space without overlapping, and is such that any one of the series is brought to coincidence with itself or some other figure of the series by any operation of  $\Gamma$ . Hence the theorem stated above is completely proved.

An interesting case is that in which  $\Gamma$  is formed by multiplying a translation-group by a point-group\* whose operations are those of the row (i).  $\Phi, \Phi_1, \Phi_2, \dots$  are then derived by transforming any one of them by the translations. We have a partitioning of space into a series of similar and similarly orientated divisions. It may be shown that each such division is, in general, in contact with fourteen others; and may therefore be supposed to have fourteen walls (not

\* That is, by combining the operations of a translation-group with those of a point-group.

in general plane). Each division must therefore 'be called a tetrakaidekahedron unless we prefer to call it a fourteen-walled cell\*.'

If  $\Gamma$  has one point-group  $G$  as subgroup, it is easily seen that it must have an infinite number of point-groups  $G, G_1, G_2, G_3, \dots$  as subgroups†. Any cell  $\phi$  is in general differently situated with regard to the elements of each of these point-groups; hence a different region  $\Phi$  is obtained from  $\phi$  by the aid of each of  $G, G_1, G_2, G_3, \dots$ . Therefore  $\Phi$  may have an infinite number of different forms corresponding to any given division of space into fundamental cells.

As an illustration we may take the division of space into fundamental cells for the group  $C_{3v}^2$ , which is shown in Fig. 186. The point of view is that of Fig. 162 (p. 221); the positions of the symmetry-elements are obvious. Each fundamental cell may be taken as a cylinder whose height is  $2\tau_z$ , and whose base is one of the numbered areas there shown.

In the following list are given a few of the infinite number of different ways of grouping the fundamental cells so as to form portions of space such as  $\Phi, \Phi_1, \Phi_2, \dots$ , corresponding to the various point-groups which are subgroups of  $C_{3v}^2$ ; those cells which are enclosed in the same brackets belong to the same portion.

Point-group.	Groups of Fundamental Cells.
$C_4$	(1, 2) (3, 4) (5, 6) (7, 8) (9, 10) .....
	or (1, 4) (2, 5) (3, 6) (7, 10) (8, 11) .....
	or (6, 1) (2, 3) (4, 5) (12, 7) (8, 9) .....
	&c., &c.
$C_3$	(1, 3, 5) (2, 4, 6) (7, 9, 11) (8, 10, 12) .....
	or (5, 27, 31) (4, 12, 32) (11, 33, 37) (10, 18, 38) .....
	&c., &c.
$C_{3v}$	(1, 2, 3, 4, 5, 6) (7, 8, 9, 10, 11, 12) (13, 14, 15, 16, 17, 18) .....
	or (4, 5, 56, 57, 42, 37) (10, 11, 62, 63, 48, 43) (16, 17, 68, 69, 54, 49) ...
	&c., &c.

If the point-group is  $C_{3v}$  we have the particular case referred to at the bottom of p. 247.

\* Lord Kelvin, Proc. Roy. Soc., lv (1894), p. 8.

† For let  $\lambda$  be the system of symmetry-elements of  $G$ , and let  $\lambda$  be brought to coincide with  $\lambda'$  by any translation of  $\Gamma$ ; then  $\lambda'$  is a system of symmetry-elements of some point-group contained in  $\Gamma$ .

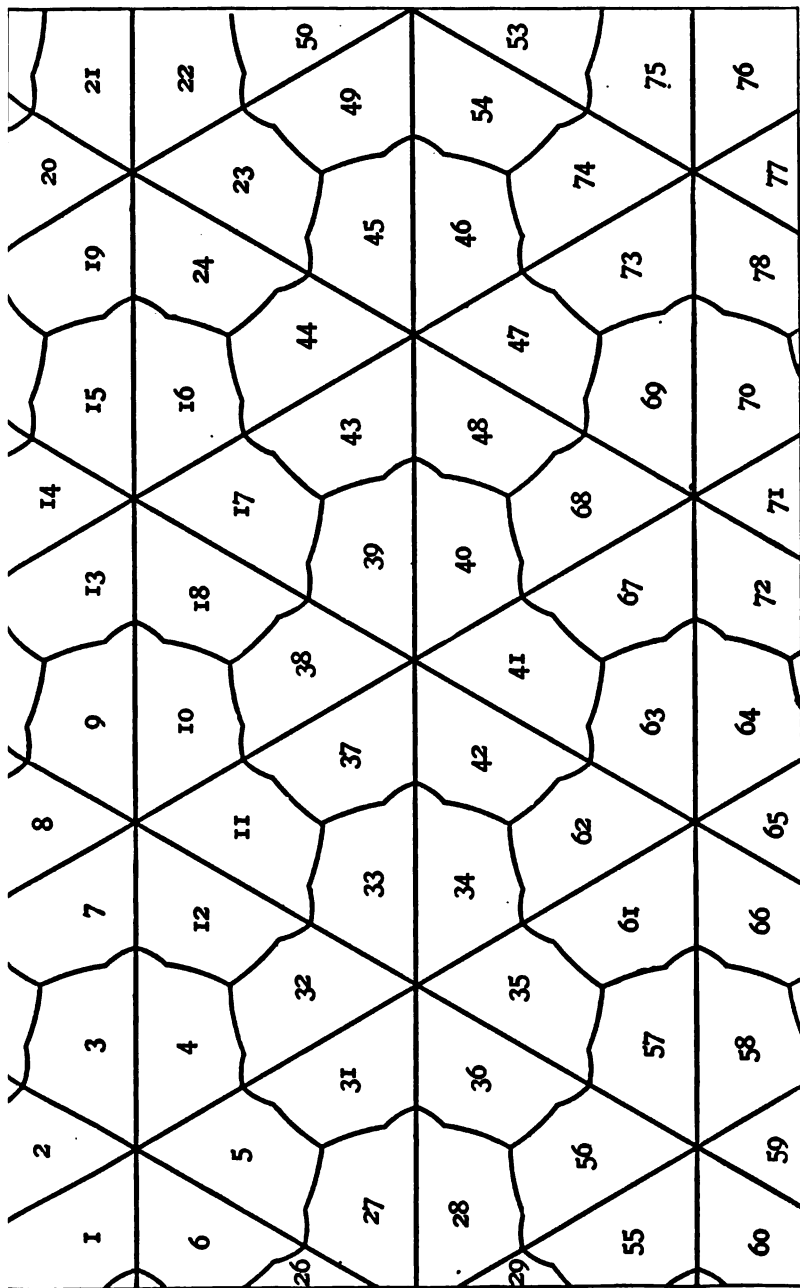


Fig. 186.

## CHAPTER XXV

## CRYSTAL-MOLECULES.

§ 1. We have shown in chapters xviii to xxiii what are the different kinds of symmetry which an infinite collection of points capable of representing the structure of a crystalline medium may have on the hypothesis of p. 118.

This hypothesis led us easily to the statement\* that such a collection is brought to self-coincidence by three independent finite translations, but by no infinitesimal translation. The most general collection of points capable of representing the structure of a crystalline medium is hence obtained by taking an arbitrary number of points, with arbitrarily chosen positions, and operating on them with all the movements (infinite in number) of one of the possible 230 space-groups. We thus obtain a collection of points whose symmetry is that of the group; and any arrangement of points which may be assumed as representing the structure of any particular crystal (provided such an arrangement has three independent finite translations and no infinitesimal translation) *must* be such a collection or a particular case of such a collection.

§ 2. Suppose that we have a crystalline medium whose physical symmetry is that of a point-group  $G$ . We shall assume that the system of molecules forming the medium has the symmetry of some space-group  $\Gamma$  isomorphous with  $G$ .

Our conception of the system of molecules is as follows: let any one molecule be represented by a cluster of points  $P, Q, R, \dots$  lying in a fundamental cell of  $\Gamma$ , then the points equivalent to  $P, Q, R, \dots$  which lie in any other fundamental cell are also supposed to represent a molecule. We have then a collection of molecules (one in each cell) which is brought to self-coincidence by every operation of  $\Gamma$ . If we suppose each fundamental cell to have only one boundary and the points  $P, Q, R, \dots$  to be close together in the cell, we can

\* For proofs of this statement starting with a rather different hypothesis, see A. Schoenflies' "Krystallsysteme und Krystallstruktur," part II, chap. xiv; K. Bohn, "Math. Annalen," liii (1900), p. 440; "Berichte der k. sächs. Gesellsch. der Wissensch. zu Leipzig," li (1899), p. 445.

form a clear mental picture of this. We may suppose, if we like (though this is quite unnecessary), that the points  $P, Q, R, \dots$  are the centres of a number of chemical atoms building up a chemical molecule, one of which is contained in each cell\*.

Two such arrangements of points, four in each cell, represented by small circles of different sizes and shading† (those of the same size and shading being equivalent to one another), are shown in Figs. 187 and 188; the symmetry is that of  $C_{2v}^2$  and  $C_{2v}^1$  respectively.

The observer is supposed to be looking in the direction  $-\tau_z$ ; the positions of the 2-al rotation-axes in Fig. 187 and of the 4-al rotation-axes in Fig. 188 are shown by crosses; the positions of the other elements of symmetry may be found from Figs. 99 and 137 (pp. 180, 207) respectively. All the circles of the same size and shading are supposed to represent points equally distant from the plane of the paper; the whole system of equivalent points is obtained by transforming those shown in the figure by the translations  $2m\tau_x$  ( $m$  a positive or negative integer).

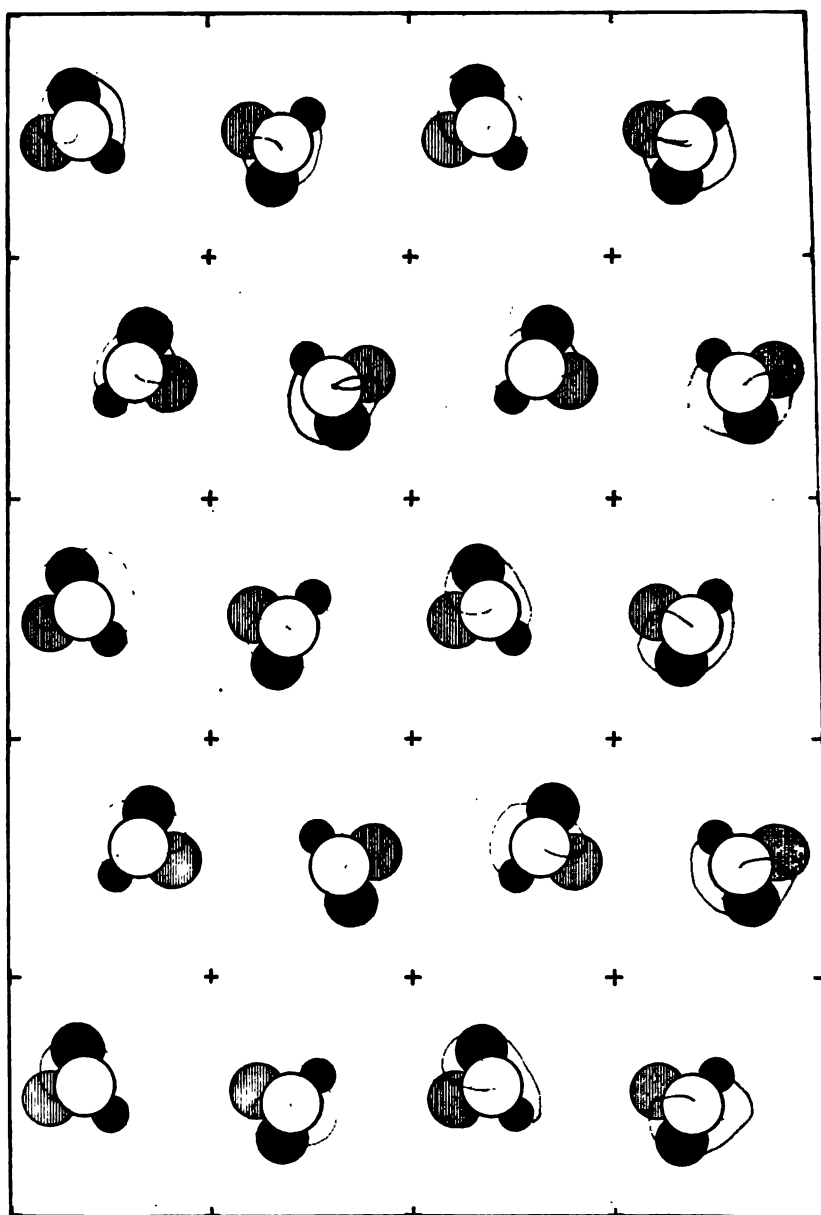
The individual molecules have no symmetry in general: they are all congruent if  $\Gamma$  has no operation of the second sort; if, however,  $\Gamma$  has such an operation the molecules may be divided into two sets, such that the molecules of either set are congruent to all the molecules of the same set and enantiomorphous to those of the other.

Some authors have objected to this division of the molecules of a crystal into two enantiomorphous sets as improbable. It is in fact in most cases possible to avoid having recourse to this division as follows:—we take (if possible) the points  $P, Q, R, \dots$  close together in a fundamental cell  $\phi$  in such positions that the equivalent points  $P_1, Q_1, R_1, \dots$  in some cell enantiomorphous to and adjoining  $\phi$  are close to  $P, Q, R, \dots$ . We then consider  $P, Q, R, \dots, P_1, Q_1, R_1, \dots$  as together representing a molecule.

Let  $P, Q, R, \dots$  be brought to coincidence with  $P_a, Q_a, R_a, \dots$ ,

\* It must be remembered that by a 'molecule' is here meant 'that portion of matter which forms a unit in building up a crystalline (or amorphous) medium.' Thus by a 'molecule of quartz' we mean the smallest portion of matter which has a right to the name 'quartz.' It does not follow that no fraction of a 'quartz molecule' has a right to the name 'silica' ( $SiO_2$ ), though it may be so.

† The points are at the centres of these circles. We may, if we prefer it, suppose the molecule represented by four spheres whose centres are these four points. Four is the smallest number of points (or spheres) which will show the difference between two enantiomorphous molecules.



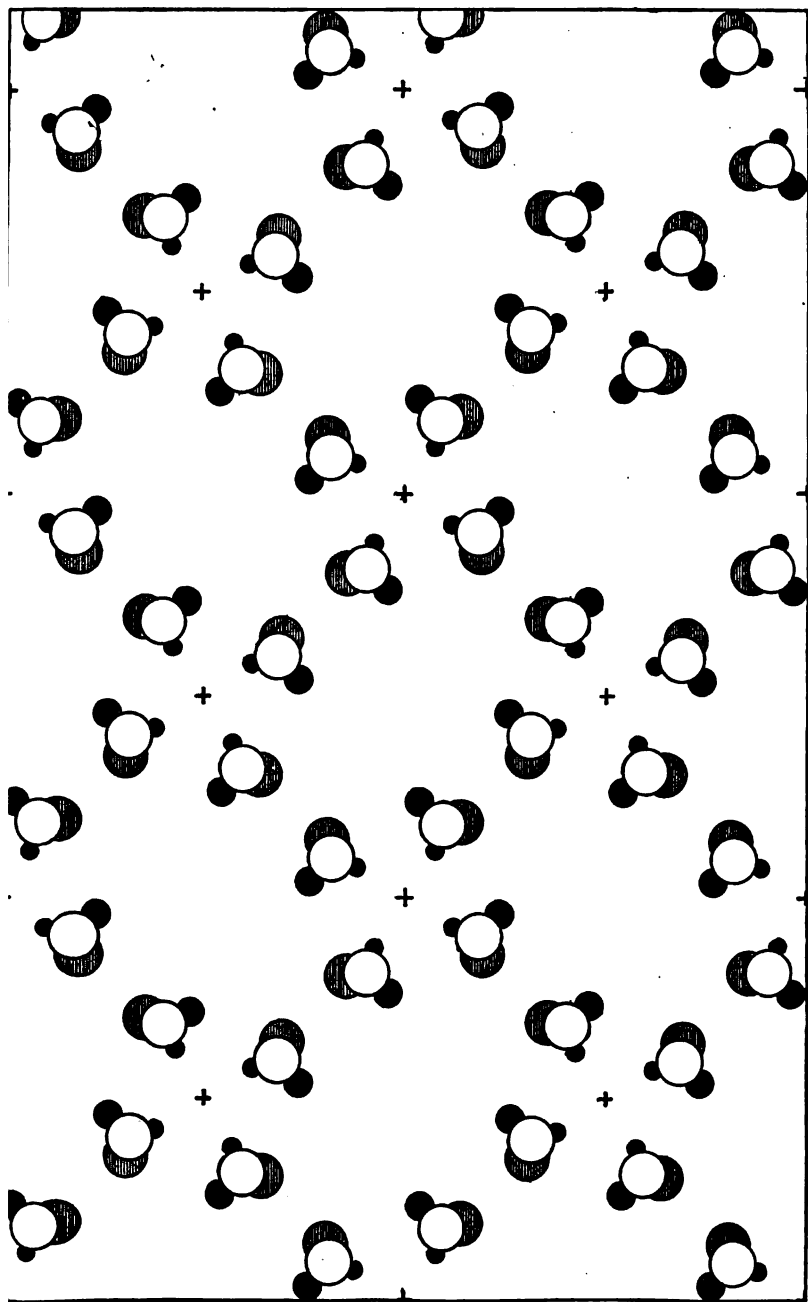


Fig. 183.

and  $P_1, Q_1, R_1, \dots$  with  $P_b, Q_b, R_b, \dots$  by an operation of the first sort contained in  $\Gamma$ ; then  $P_a, Q_a, R_a, \dots, P_b, Q_b, R_b, \dots$  together represent a molecule congruent to  $P, Q, R, \dots$ ,  $P_1, Q_1, R_1, \dots$ , and proceeding in this way we can group all the points equivalent to  $P, Q, R, \dots$  into a series of *congruent 'molecules.'* The weakness of this method lies in the fact that it cannot be used to explain the structure of crystals of *every* class, without supposing that these congruent molecules may interlace with one another, an idea which hardly commends itself to a physicist; the method fails, for instance, in the case of  $C_{3d}^m$ .

§ 3. Now suppose that the group  $\Gamma$  can be derived by multiplying a translation-group by a *point-group* whose operations leave a point  $A$  unmoved. Let  $\phi$  be a fundamental cell whose boundary passes through  $A$ , and suppose that  $\phi$  is brought to coincide with  $\phi, \phi_1, \phi_2, \dots, \phi_m$  by the operations of the point-group. All other fundamental cells are obtained by transforming these by the translations.

Consider points  $P, Q, R, \dots$  in  $\phi$  close to  $A$ , and let the points equivalent to them in  $\phi_1, \phi_2, \dots, \phi_m$  be

$$P_1, Q_1, R_1, \dots; P_2, Q_2, R_2, \dots; \dots; P_m, Q_m, R_m, \dots;$$

all these points are evidently close to  $A$ .

Then

$$P, Q, R, \dots; P_1, Q_1, R_1, \dots; P_2, Q_2, R_2, \dots; \dots; P_m, Q_m, R_m, \dots;$$

may be considered as together forming a small 'molecule'  $\mu$  close to  $A$ ; this molecule has obviously the symmetry of the point-group.

Evidently in the same way all other points equivalent to  $P, Q, R, \dots$ , may be grouped into 'molecules' derived from  $\mu$  by transforming by the translations of the group.

This is Bravais' conception of a series of molecules each with the symmetry of a point-group, whose centres are arranged at the points of a lattice representing a translation-group. His theory has now been cleared from the accusation that it only defers the difficulty of explaining the symmetry of a crystal, and that it gives no adequate explanation of the assumed symmetry of the individual molecules (p. 143).

It has in fact been shown that, if his 'molecules' may be considered composite\*, his structures are included among the most general arrangements of particles possible. His method has, however, the disadvantage that it is somewhat arbitrary;

\* Bravais himself makes this assumption; see p. 143.

for he only makes use of a limited number of groups of movements, namely, the seventy-three which are obtained by multiplying a translation-group by a point-group; such as  $Q^1, Q^6, Q^7, Q^8*$ .

In this and in the previous section we have, for the sake of clearness, supposed our points  $P, Q, R, \dots$  finite in number and close together in a fundamental cell; this is not necessary, and we may take *any* points in a fundamental cell, and all points equivalent to them as representing the structure of a crystalline medium. The extreme cases are, on the one hand the case of a single point in each cell, on the other the case of an infinite number of points in each cell completely filling the cell. If we consider points infinite in number and completely filling a fundamental cell as together constituting a physical unit in making up the structure, we must take the cell with a single boundary; for we can hardly consider *physical* units as interlacing.

§ 4. It will be readily understood that our conception of crystal-structure is in accordance with the observed properties of matter, when we remember that our methods of observation do not enable us to determine the physical properties of a substance along a mathematical line, but only to determine the sum total of these properties along all the lines in a given direction which meet an area whose dimensions are large compared with the distances between neighbouring molecules. We should therefore expect, assuming the above theory of crystal-structure, that the physical properties in a fixed direction of all parts of a homogeneous crystalline medium would appear to be the same in spite of the fact that the material is coarse-grained. These considerations apply to an amorphous medium; but, while the geometrical symmetry of a crystalline medium would lead us to expect that its physical properties are different in different directions, there is no reason why we should anticipate this difference in the case of a medium which lacks this geometrical symmetry; we should therefore expect the physical properties of a homogeneous amorphous solid to be the same in every direction.

Since all the operations of any space-group are obtained by multiplying a finite number† of them by the translations (p. 158), therefore all the points of an equivalent system are

\* Obtained by multiplying  $Q$  by  $\Gamma_0, \Gamma'_0, \Gamma''_0, \Gamma'''_0$  respectively.

† This number is equal to the number of operations  $m$  in the isomorphous point-group.

obtained by transforming a finite number of them by the translations. This applies to the system equivalent to each of the points  $P, Q, R, \dots$  of § 2. Hence the series of points representing the constituent parts of a crystalline solid are the points of a series of similar and similarly orientated lattices\* (any one of which represents the translations of the group).

We may therefore apply without appreciable modification Bravais' explanation of the cleavage, the occurrence of crystal faces, the law of rational indices, &c., as given in chapter xv.

§ 5. It must be remembered that all we have done is to obtain every structure which is *geometrically* possible, on the assumption that the structure has three independent finite translations, and no infinitesimal translation. It does not follow that every such structure is *mechanically* possible; that if particles were arranged in the way suggested by the geometrical investigation they would be in stable equilibrium. The geometrical theory is now complete, the mechanical theory is still in its infancy. That some particular cases of the general geometrical structure (if not the most general structure itself) are mechanically possible may be safely prophesied; further than this it is not at present possible to go†.

If the general symmetry of a crystal is that of a point-group  $G$ , its intimate structure has the symmetry of some space-group isomorphous with  $G$ ; but it cannot yet be determined which of these isomorphous space-groups represents the intimate symmetry of *any* given material.

Attempts have been made to solve this problem in a few special cases; thus Viola assigns  $D_{3d}^2$  to Calcite and  $C_{3h}^2$  to Aragonite‡. Fedorow allots  $O_h^2, T_d^4, T_h^2$  at high temperatures, and  $D_{4h}^4, D_{2d}^1, Q_h^2$  at ordinary temperatures to Leucite, Boracite, and Perowskite respectively. He bases this allotment on the optical behaviour of the substances, but further evidence is needed.

If we could find a crystal whose symmetry is that of  $C_{3h}$ , we could allot the space-group  $C_{3h}^1$  to it at once, since there is no other space-group isomorphous with  $C_{3h}$ ; such a crystal, however, has not yet been found.

Crystals belonging to an enantiomorphous hemihedry can crystallize in two enantiomorphous forms (p. 92). In some cases two plates of similar direction and thickness cut from two

\* Their number is  $mn$ , where  $n$  is the number of the points  $P, Q, R, \dots$

† See Lord Kelvin's paper in the "Phil. Mag.," 1902, p. 149.

‡ "Zeitschr. f. Kryst. u. Min.," xxviii, p. 230.

such enantiomorphous crystals rotate the plane of polarized light through equal and opposite angles. It is natural to allot to two such enantiomorphous forms space-groups which are similar, except that if one has left-handed screws the other has right-handed, and vice versa \*. For example, it is natural to assign to the two varieties of quartz, either the two groups  $D_3^3$  and  $D_3^5$ , or else the two groups  $D_3^4$  and  $D_3^6$ ; Sohncke has given reasons for adopting the latter alternative.

To crystals which rotate the plane of polarization and which belong to the regular enantiomorphous hemihedry we should allot the groups  $O_6$  and  $O_7$ ; no crystals, however, have yet been found belonging to this class which rotate the plane of polarization.

Among recent attempts to obtain the exact structure of certain crystals (not merely to assign to them the appropriate space-group) may be mentioned those of Barlow † and Sollas ‡. Both of these authors represent all atoms of the same element in a given crystal by spheres of the same size. The former chooses arrangements (consistent with the known physical symmetry of the crystals) in which the spheres are packed together as closely as possible. Professor Sollas, on the other hand, is guided by the observed density (and symmetry) in his selection of suitable arrangements. In the absence of any sufficiently developed mechanical theory which might demand closest packing, the latter method is perhaps to be preferred. Of course any one of the collections of spheres described by either author is brought to self-coincidence by every operation of one of the 230 possible space-groups.

\* This is only possible with optically isotropic and uniaxial crystals. The suggestion cannot be applied to biaxial crystals, which rotate the plane of polarization, such as cane-sugar and rochelle salt ("Phil. Mag.," 1901, ii, p. 361); and the rotation in such cases must be ascribed to the constitution of the individual molecules.

† "Scientific Proceedings of the Royal Dublin Society," vol. viii, part vi, No. 62 (1897).

‡ "Proc. Roy. Soc.," lxiii, lxix.

## CHAPTER XXVI

## HISTORY OF THE STRUCTURE-THEORIES.

§ 1. A clear and full account of the earlier attempts to suggest an explanation of the internal structure of crystals is given in L. Sohncke's "Entwicklung einer Theorie der Krystallstruktur." Bravais' theory, which was a great step in advance, was put forward in 1850\*. In 1869 C. Jordan published a note 'Sur les groupes de mouvements' †, in which he gives a full and complete proof of the existence of those groups which contain only operations of the first sort. He gives groups containing infinitesimal translations, and others involving only one or two independent translations, and it is by no means certain (though it is possible ‡) that he realized the crystallographic application of his work. L. Sohncke, in 1879 §, first clearly pointed out this application; his book deals only with the sixty-five groups which contain no operations of the second sort, and which have three independent and no infinitesimal translations.

At first he represented the structure of crystalline media by a system of points equivalent to a single point *P*. By considering structures, in which *P* has some special position with regard to the axes of symmetry, he was able to account for crystals whose physical symmetry is that of point-groups containing operations of the second sort. Later on he found this device unsatisfactory, and for the single point *P* he substituted *n* points ||; thus arriving practically at that method of representing the structure of a crystal by a system of *congruent* molecules which has been described on p. 251. To this method he adhered, for he considered Schoenflies' division of the molecules of a substance into two enantiomorphous sets extremely improbable.

On the *probability* of either method it is impossible to form an opinion in the absence of any mechanical theory; and

\* Mémoire sur les systèmes formés par des points, &c., "Journ. de l'École Polyt.," xix, 88, p. 1.

† "Ann. di matematica pura ed applicata," ii, 2.

‡ His use of the terms 'merohedry,' &c., may be a sign that he had this application in view.

§ "Entwicklung einer Theorie der Krystallstruktur."

|| Erweiterung der Theorie der Krystallstruktur, "Zeitschr. f. Krystall. u. Min.," xiv, p. 485.

Schoenflies' contention that it is sufficient if his procedure is geometrically possible seems reasonable. Moreover, Sohncke's arguments were greatly weakened by the fact that he could not apply his method in every case (see p. 254).

The proof of the existence of all the 230 groups of movements is due to Fedorow, Schoenflies, and Barlow. Fedorow's and Schoenflies' work appeared about 1890; the former had a slight priority, but his work was published in Russian, and was, therefore, not so widely circulated as it might otherwise have been. He divided the 230 groups into 73 which are obtained by multiplying a translation-group by a point-group, 54 whose operations of the first sort only can be so derived, and 103 whose operations of the first sort cannot be so derived\*.  $Q_h^1$ ,  $Q_h^2$ , and  $Q_h^6$  are examples of these three classes respectively.

Schoenflies published preliminary notes of his work in the "Math. Annalen"†, and in 1891 produced his "Krystallsysteme und Krystallstruktur," in which he gives a full description of the groups, and the method of dividing space into 'fundamental cells.' We have in this book chosen lines of proof which are, in most cases, practically identical with those of Schoenflies.

Barlow published his work later‡ but independently. He gives first those 65 groups which have only operations of the first sort (Sohncke's groups); then the 92 groups derived from them by multiplying by an inversion; then those 71 groups of the second sort which cannot be derived with the aid of an inversion, but which can be derived by multiplying by an operation isomorphous with a reflexion; and, lastly, those two groups which can only be derived by multiplying by a rotatory-reflexion ( $C_4^1$  and  $C_4^2$ ). He confines himself to a statement of the results arrived at without giving a full account of the methods employed§.

The geometrical theory of crystal-structure seems to be now fairly complete; it is probable that further advance is to be expected on the physical or mechanical side.

For a fuller historical account the reader is referred to the above quoted work of Sohncke, and to the British Association report on "The Structure of Crystals," 1901.

\* He calls these groups symmorphous, hemi-symmorphous, and asymmorphous respectively.

† "Math. Ann.," xxviii, xxix, xxxiv.

‡ "Zeitschr. f. Kryst. u. Min.," xxiii, 1894.

§ For a comparison of the results of Fedorow, Schoenflies, and Barlow see "Phil. Mag.," Feb. 1902.

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THE END



